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Notes and Correspondence
On the consistency between dynamical and thermodynamic equations with prescribed vertical motion in an analytical tropical cyclone model

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In our earlier paper, we presented an analytical model for intensifying tropical cyclones (TCs), in which the effects of all nonlinear terms in the horizontal momentum equations are retained. This analytical model was obtained by prescribing the time evolution of the vertical motion. In this paper, we demonstrate that, with the prescribed vertical motion, the system of governing equations for geophysical flows can be separated into two subsystems: one consisting of the horizontal momentum and continuity equations, and the other including the vertical momentum and thermodynamical equations. Results show that the analytical solutions for the horizontal winds obtained from the first subsystem are consistent with the mass field in the second subsystem, such as the thermal wind relationship in TCs. Furthermore, we show that use of any functional form for the mean vertical motion will not affect our previous major conclusions about the different growth rates either between the secondary and primary circulations or between the inner and outer regions in TCs. Copyright © 2010 Royal Meteorological Society

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1. Introduction

Although the forecasts of tropical cyclone (TC) tracks have achieved considerable progress, forecasting TC intensity changes, particularly for rapidly developing TCs, is still a challenging issue. This could be attributed partly to the lack of understanding of the nonlinear feedback processes involved in the genesis and subsequent more rapid intensification of TCs. Theoretical models for TC intensification used to be formulated from the perspective of the secondary circulation (SC; e.g. Charney and Eliassen, 1964; Yanai, 1964). However, little is known about the relationship between the growth of the SC and the intensity and intensity changes of the primary circulation (PC), which are often represented by the maximum surface winds, $V_{\max}$, and the minimum central pressure or the central pressure drop, $\delta P_{\min}$.

Recently, we developed an analytical model (Kieu and Zhang, 2009; hereafter KZ09), in which the effects of all nonlinear terms in the horizontal momentum equations are retained, for the intensifying stage of TCs. KZ09 has shown that the growth of the PC is fundamentally different from that of the SC. In addition, the growth rate of the PC in the outer region differs from that in the inner-core region. Several other important conclusions have also been obtained, including the dependence of the TC growth on its vertical structure as well as the bottom upward development of the PC. A key procedure to obtaining the analytical solutions for the PC is to assume an explicit time-dependent solution for the area-averaged or mean vertical motion $w(t)$ such that the complicated thermodynamic processes, including surface latent heat fluxes, could be bypassed and the number of governing equations for TC flows can be reduced. This assumption is based on the fact that deep convection tends to be more organized in the inner-core region as a TC transitions from a depression to a hurricane phase.

In their comments, Montgomery and Smith (2010, hereafter MS10) raised the following three issues on our analytical model:

(i) our assumption of the exponential growth rate for the mean vertical motion $\overline{w}(t)$ is not supported by observations;
(ii) the analytical solutions do not satisfy the vertical momentum and thermodynamical equations; and
(iii) our analytical model neglects the contribution of warm core to the central pressure drops.

For the sake of our discussions, the last two issues will be combined into one issue involving the consistency between the dynamical and thermodynamical equations in our analytical model. Thus, MS10 concluded that our analytical model is not relevant to understanding the intensification of TCs. We believe that MS10 have misinterpreted the essence of our results. Here we would like to take this opportunity to clarify some issues associated with our analytical solutions and provide point-by-point replies to MS10's comments.

2. Exponential growth of the mean vertical motion

It should be first mentioned that although vertical motion can be directly measured by VHF Doppler radar, we have not seen direct observations of the vertical motion on the TC scale in the literature. Nevertheless, the vertical motion field generated by today's cloud-resolving models, when estimated meaningfully, has been widely regarded as a good proxy to represent the vertical overturning in TCs. For this reason, we formulated the evolution of the mean vertical motion using the cloud-resolving simulation of hurricane *Wilma* (2005) during its intensifying stage (KZ09). To characterize this time evolution in our analytical model, we defined $\alpha = w(t)/w\ (t = 6\text{ h})$ as the ratio of the evolving mean vertical motion to its value at 6 h into the simulation, which is apparently dimensionless. (The value of $w\ (t = 6\text{ h}) = 0.12\text{ m s}^{-1}$ was provided in KZ09 but we regret that it was not given in the caption of Figure 1(a) therein.) The growths of $V_{\max}$ (t) and $\delta P_{\min}$ (t) with respect to their $t = 6$ h values are also given in the same figure as $w(t)$ in order to show their different growing characteristics. While the growth of the mean vertical motion $w(t)$, given in Figure 1(a) of KZ09, is more or less linear in time, i.e. $w(t)/w_0 \approx (1 + \beta t)$, where β is the growth rate of the vertical motion, it is mathematically more convenient to express it in an exponential form so that taking derivatives and integrations becomes simple. This approach has also been widely used in various theoretical models to characterize the growth of atmospheric perturbations (e.g. Charney and Eliassen, 1964; Yanai, 1964; Holton, 1992; Ooyama, 1969).

The main goal of KZ09's study is to see *how the PC evolves with time if the SC could grow exponentially as previously suggested*. Indeed, it can be shown that the use of any functional form for the evolution of the vertical motion does not change the key results about the different growth rates between the SC and PC obtained in our analytical model. Specifically, if one assumes that the time evolution of the vertical motion is expressed in the form of $f(t)$, where $f(t)$ is an arbitrary function of time, the temporal dependence of the tangential flow will be then proportional to $\exp\{f(t)\}$. For example, if $f(t)$ is chosen to be $\exp(\beta t)$ as in previous theories, the tangential wind will grow as $\exp\{\exp(\beta t)\}$. If the linear function $f(t) = 1 + \beta t$ is used to better fit the evolution of w, then the tangential wind will evolve as $\exp(1 + \beta t)$. Thus, we may state that any functional form of $f(t)$ does not alter the fact that the primary and secondary circulations grow at different rates, which is one of the key issues addressed in KZ09.

MS10 argued that the mean vertical motion would not increase with time since convective updraughts would become less intense due to both the increasing stiffness of a TC vortex and the stabilizing effects of the upper-level warming. While this may be true, as TCs will keep intensifying forever otherwise, such stabilizing effects are expected to occur toward the end of rather than during the intensification stage. As emphasized in KZ09, our analytical model is intended only for the RI stage rather than the life cyle of TCs. We have examined several model-simulated TCs, and the results do confirm that the area-averaged vertical motion increases with time during the intensifying stage of the TCs (not shown). Intuitively, more rapid intensification should correspond to stronger low-level convergence (i.e. radial inflow), and more intense upward motion in the eyewall or storm-scale vertical motion.

3. Consistency between the dynamical and thermodynamic equations

MS10 correctly pointed out that the thermodynamic and vertical momentum equations were not explicitly included in deriving our analytical solutions. However, they jumped simply to conclude that our solutions do not satisfy these two equations and neglect the contribution of warm core to the central pressure drops. We will now show that this is not the case. Let us start from the vertical momentum equation (3) of KZ09 in the cylindrical (r, z) coordinates, following Willoughby (1979),

$$b = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} + \frac{\partial \phi}{\partial z}, \qquad (1)$$

where u is the radial flow, $b = g(T - \overline{T})/\overline{T}$ is the buoyancy, g is gravity, $\overline{T}$ is the mean temperature of the far environment, and ϕ is the geopotential height perturbation. Note that the frictional term has been neglected herein for the convenience of our subsequent discussions. Given the analytical solutions of $u(r,z,t)$, $w(r,z,t)$, and $\phi(r,z,t)$ from KZ09, Eq. (1) gives an explicit solution for the buoyancy $b(r,z,t)$, as shown in Figure 1 in a radial–height cross-section. A warm core down to the surface is evident in accordance with the lower pressure in the core region, except that its peak is located at a too low altitude due to the use of the sine function for $w(r,z,t)$. The presence of the warm core is also consistent with decreases of the rotational flows with height (Figure 8

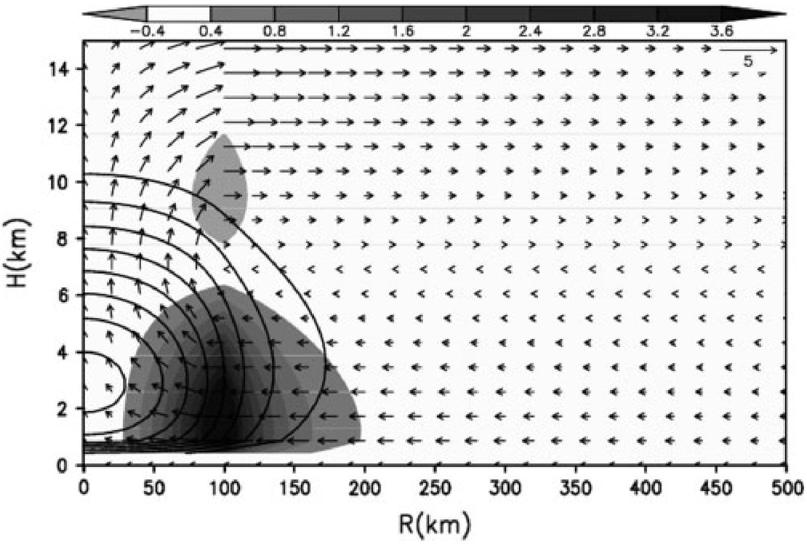

Figure 1. The radial–height cross-section of the buoyancy (contoured at intervals of 0.01 m s^{-2}) and the diabatic heating rate (shaded at intervals of 0.4 K day^{-1}) that are calculated from Eqs (1) and (2), respectively, using the same data as in KZ09, including the superimposed in-plane flow vectors. Note that the vertical motion has been multiplied by a factor of 5 for illustration purposes.

in KZ09), thus ensuring the validity of the thermal wind relationship. Because the most weighted contribution to the buoyancy is related to the lower pressure or *vice versa*, i.e. the last term on the r.h.s. of Eq. (1), a warm core and central pressure drop are highly correlated. Thus, the 'physical inconsistency' between the warm core and the prescribed upward motion in the core region, as pointed out by MS10, is not warranted.

We next show that our analytical solutions are also consistent with the thermodynamical equation (5) in KZ09, which is given as

$$J = \frac{\overline{T}}{g}\left(\frac{\partial b}{\partial t} + u\frac{\partial b}{\partial r} + N^2 w\right), \tag{2}$$

where J is the diabatic heating rate and N is the static stability. With the solutions of $u(r,z,t)$, $w(r,z,t)$, and $b(r,z,t)$ known, and assuming that the basic state does not change substantially during the intensification period, Eq. (2) determines the heating rate $J(r,z,t)$ consistently with the buoyancy and vertical motion. For different reasons, Anthes (1982), Puri and Miller (1990), and Mapes and Houze (1995) have treated or diagnosed the vertical motion as a proxy for diabatic heating in TCs and mesoscale convective systems. Because the radial gradient of buoyancy b is maximized near r = 50 km, one can see from Figure 1 that the maximum heating rate is located about 100 km from the centre. The lower-than-expected level of the maximum heating rate is attributable to the lower-level warm core which is in turn related to the prescribed half-sine harmonics of the mean vertical motion. Although a better approximation of the vertical profile for $w(t)$ could relieve such a lower-level characteristic of the heating profile, the consistency between the dynamical and thermodynamical equations is clearly valid when the vertical motion field is known. Depending on the competition between the local change and the radial advection of buoyancy, i.e. the first and second terms on the r.h.s. of Eq. (2), the radius of the maximum heating rate will vary with time. With the buoyancy b and diabatic heating $J(r,z,t)$ obtained above, our analytical solutions thus satisfy both the vertical momentum and thermodynamical equations, and there is no violation of either Newton's laws or the thermodynamic law as MS10 claimed. More details can be found in Kieu (2008).

Note that since $b(r,z,t)$ shares the same time dependence as the geopotential perturbation ϕ, according to Eq. (1), the diabatic heating J will also increase at the double exponential rate. Such an increase is expected for intensifying TCs because of the positive feedback between either the friction-induced moist convergence and diabatic heating (Charney and Elliasen, 1964) or surface heat exchange and diabatic heating (Emanuel, 1986). Of course, this feedback process will not continue forever. As the thermodynamic energy reaches the maximum potential capability controlled by the sea surface temperature, the outflow upper temperature, and available moisture for condensation, the feedback will be limited and TCs will not intensify further. This puts an upper bound on the values of the TC maximum potential intensity (Emanuel, 1986). Imposing this upper bound of maximum intensity, it is possible to obtain an upper limit for the intensification *rate* in our model.

Recall that the upward motion in the core region as seen in Figure 1 is due to the assumption of the top-hat radial function for the vertical motion, i.e.

$$w(r,z,t) = \begin{cases} w_0 \sin(\lambda z)\exp(\beta t) & r \le a, \\ 0 & r > a, \end{cases} \tag{3}$$

where a is a radius characterizing the horizontal scale of a TC, $\lambda = \pi/H_0$ is a constant representing the inverse of the tropospheric depth, and w_0 is the maximum vertical motion at the mid-level. Storm-scale upward motion should be expected for a well-developed TC with strong ascending motion in the eyewall and subsidence in the eye (Liu *et al.*, 1999), when it is area-averaged within the radius of $r = a$. Although the top-hat profile (3) could not capture realistically the detailed subsidence within the eye, the use of the analytical solutions derived from (3) to construct an initial bogus hurricane vortex for modelling studies does not prevent such subsidence at the later time. Zhang *et al.* (2000) showed that rapid rotation in the eyewall tends to produce a downward vertical pressure gradient force to compensate for the net positive buoyancy in the eye. Thus, the subsidence will develop in the eye soon after the wind field is adjusted to the mass field. This scenario could be seen from Xiao *et al.* (2000), who showed higher specific humidity at the model initial time at the centre of a bogus hurricane vortex but drier air with the subsidence in the core region after 49 h into the model integration (Figures 4 and 9 of Xiao *et al.*, 2000). Because of the above results, our analytical solutions could be used to provide a bogus axisymmetric vortex, as given in Figure 8 of KZ09, for TC models.

It should be mentioned that the above procedures to obtain the diabatic heating function *J(r, z, t)* from a given vertical motion field in TCs are opposite to the traditional approach in which the transverse circulations are treated as a response to diabatic heating through the well-known Sawyer–Eliassen equations (e.g. Yanai, 1964; Schubert *et al.*, 2007). In this quasi-balanced system, one has to assume *a priori* the vertical structure of the PC before the Sawyer–Eliassen equation can actually be solved. In our approach, we proceed with the mean vertical motion field, and derive the vertical structure of the PC. In some sense, our approach has certain advantages over the traditional one as the vertical structures of the vertical motion are often better known than those of the PC. That is, in many cases, the vertical distribution of the vertical motion may be approximated by half sine harmonics, as shown in KZ09. In their case, Mapes and Houze (1995) felt that diagnosing the vertical motion instead of diabatic heating 'avoids the loss of information between heating profile measurement studies on the one hand and assessments of the large-scale impacts of heating profiles on the other'.

4. Concluding remarks

In this reply, we have argued, based on the model-simulated TCs, that the mean vertical motion tends to increase with time during the intensification stage. The increasing rate, though linear, could be approximated as an exponential form just for the sake of mathematical derivations. In particular, we have shown that, by prescribing the vertical motion field *w(r,z,t)*, the governing equations system for TCs, i.e. Eqs (1)–(5) in KZ09, could be separated into two subsystems: one consisting of the horizontal momentum and continuity equations as used in KZ09, and the other including the vertical momentum and thermodynamical equations as presented herein. Because of this finding, we are able to obtain the relationship between the SC and PC in KZ09 without invoking complicated thermodynamic processes. Thus, we may conclude that the use of the prescribed vertical motion is consistent with both the vertical momentum and thermodynamical equations, including the thermal wind relationship between the warm core in the eye and the upward decreased tangential winds in the eyewall.

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