Derivation of QG omega equation

Start by considering the QG vorticity equation and QG thermodynamic energy equations (ignoring diabatic effects):

$$\frac{\partial \zeta_g}{\partial t} = -\vec{V}_g \cdot \nabla\left(\zeta_g + f\right) + f_o \frac{\partial \omega}{\partial p} \tag{1}$$

and

$$\frac{\partial}{\partial t}\left(-\frac{\partial \phi}{\partial p}\right) = -\vec{V}_g \cdot \nabla\left(-\frac{\partial \phi}{\partial p}\right) + \sigma\omega \tag{2}$$

Rearrange the derivatives on the LHS of (2) to put in terms of geopoential tendency:

$$-\frac{\partial}{\partial p}\left(\frac{\partial \phi}{\partial t}\right) = -\vec{V}_g \cdot \nabla\left(-\frac{\partial \phi}{\partial p}\right) + \sigma\omega \tag{3}$$

Consider that vorticity can be rewritten as the Laplacian of the geopotential

$$\zeta_g = \frac{1}{f_o}\nabla^2\phi \tag{4}$$

we can rewrite the QG vorticity equation (1) as:

$$\frac{1}{f_o}\frac{\partial}{\partial t}\left(\nabla^2\phi\right) = -\vec{V}_g \cdot \nabla\left(\zeta_g + f\right) + f_o \frac{\partial \omega}{\partial p} \tag{5}$$

changing the order of derivatives in the LHS

$$\frac{1}{f_o}\nabla^2\frac{\partial \phi}{\partial t} = -\vec{V}_g \cdot \nabla\left(\zeta_g + f\right) + f_o \frac{\partial \omega}{\partial p}\,. \tag{6}$$

Next, notice that both the modified vorticity equation (6) and modified energy equation (3) have geopotential tendency terms. We will take advantage of this with some minor manipulation in order to combine these two equations. To do so, we will take the Laplacian (∇^2) of (3) and take ($-f_o\frac{\partial}{\partial p}$) of (6):

$$\nabla^2\left(-\frac{\partial}{\partial p}\left(\frac{\partial \phi}{\partial t}\right) = -\vec{V}g \cdot \nabla\left(-\frac{\partial \phi}{\partial p}\right) + \sigma\omega\right) \tag{7}$$

$$-f_o\frac{\partial}{\partial p}\left(\frac{1}{f_o}\nabla^2\frac{\partial \phi}{\partial t} = -\vec{V}_g \cdot \nabla\left(\zeta_g + f\right) + f_o \frac{\partial \omega}{\partial p}\right) \tag{8}$$

Expand out so that we can identify the common term that can then be combined:

$$\underline{\nabla^2\left(-\frac{\partial}{\partial p}\left(\frac{\partial \phi}{\partial t}\right)\right)} = \nabla^2\left(-\vec{V}g\cdot\nabla\left(-\frac{\partial \phi}{\partial p}\right)+\sigma\omega\right) \tag{9}$$

$$\underline{\nabla^2\left(-\frac{\partial}{\partial p}\left(\frac{\partial \phi}{\partial t}\right)\right)} = -f_o\frac{\partial}{\partial p}\left(-\vec{V}_g\cdot\nabla\left(\zeta_g+f\right)+f_o\frac{\partial \omega}{\partial p}\right). \tag{10}$$

(10):

$$\nabla^2\left(-\vec{V}g\cdot\nabla\left(-\frac{\partial \phi}{\partial p}\right)+\sigma\omega\right) = -f_o\frac{\partial}{\partial p}\left(-\vec{V}_g\cdot\nabla\left(\zeta_g+f\right)+f_o\frac{\partial \omega}{\partial p}\right) \tag{11}$$

Multiply out to get:

$$\nabla^2\left(-\vec{V}g\cdot\nabla\left(-\frac{\partial \phi}{\partial p}\right)\right)+\nabla^2\left(\sigma\omega\right) = f_o\frac{\partial}{\partial p}\left(\vec{V}_g\cdot\nabla\left(\zeta_g+f\right)\right)-f_o^2\frac{\partial^2 \omega}{\partial p^2} \tag{12}$$

Moving the ω terms to the LHS the other terms to the RHS:

$$\nabla^2\left(\sigma\omega\right)+f_o^2\frac{\partial^2 \omega}{\partial p^2} = \nabla^2\left(\vec{V}g\cdot\nabla\left(-\frac{\partial \phi}{\partial p}\right)\right)+f_o\frac{\partial}{\partial p}\left(\vec{V}_g\cdot\nabla\left(\zeta_g+f\right)\right) \tag{13}$$

Recalling that we have assumed that the static stability is horizontally homogeneous, divide the entire equation by (σ) to get

$$\nabla^2\omega+\frac{f_o{}^2}{\sigma}\frac{\partial^2 \omega}{\partial p^2} = \frac{1}{\sigma}\nabla^2\left(\vec{V}g\cdot\nabla\left(-\frac{\partial \phi}{\partial p}\right)\right)+\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left(\vec{V}_g\cdot\nabla\left(\zeta_g+f\right)\right) \tag{14}$$

And finally, combining the LHS to arrive at the QG omega equation:

$$\left(\nabla^2+\frac{f_o{}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = \frac{1}{\sigma}\nabla^2\left(\vec{V}g\cdot\nabla\left(-\frac{\partial \phi}{\partial p}\right)\right)+\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left(\vec{V}_g\cdot\nabla\left(\zeta_g+f\right)\right) \tag{15}$$

Conceptually, it is sometimes easier to add minus signs to the RHS terms in order to think in terms of advection:

$$\left(\nabla^2+\frac{f_o{}^2}{\sigma}\frac{\partial^2}{\partial p^2}\right)\omega = -\frac{1}{\sigma}\nabla^2\left(-\vec{V}g\cdot\nabla\left(-\frac{\partial \phi}{\partial p}\right)\right)-\frac{f_o}{\sigma}\frac{\partial}{\partial p}\left(-\vec{V}_g\cdot\nabla\left(\zeta_g+f\right)\right) \tag{16}$$

In summary:
Upward vertical motion (negative omega) can be forced by a local maximum in thermal advection or positive differential vorticity advection. This is purely a diagnostic equation to evaluate forcing for omega.