Vorticity Equation (AOSC470/600, Prof. Kleist)

Starting with the notion that we have an expression for the relative vorticity:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

We need to take the x derivative of the meridional momentum equation and y derivative of the zonal momentum equation (Martin 5.31a and 5.31b) to try and get terms that look like relative vorticity, i.e.:

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}\right)$$
$$-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}\right)$$

To compute the derivatives, all of the terms (except d/dt in each) require the use of the product rule for differentiation (quotient rule for last term of each expression). Doing so results in the following expanded derivatives of the momentum equations:

$$\left[\left(\frac{\partial}{\partial x}\frac{\partial v}{\partial t}\right) + \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2}\right) + \left(\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + v\frac{\partial^2 v}{\partial x \partial y}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} + w\frac{\partial^2 v}{\partial x \partial z}\right) + \left(f\frac{\partial u}{\partial x} + \cancel{u\frac{\partial f}{\partial x}}\right) = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y} + \frac{1}{\rho^2}\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}\right]$$
$$-\left[\left(\frac{\partial}{\partial y}\frac{\partial u}{\partial t}\right) + \left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + u\frac{\partial^2 u}{\partial x \partial y}\right) + \left(\frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + v\frac{\partial^2 u}{\partial y^2}\right) + \left(\frac{\partial w}{\partial y}\frac{\partial u}{\partial z} + w\frac{\partial^2 u}{\partial y \partial z}\right) - \left(f\frac{\partial v}{\partial y} + v\frac{\partial f}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y} + \frac{1}{\rho^2}\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right]$$

We can eliminate the term that involves a zonal derivative of Coriolis, and then combine the two equations into a single expression (boxes will now be used in grouping terms together):

$$\boxed{\left(\frac{\partial}{\partial x}\frac{\partial v}{\partial t}\right)} + \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \boxed{u\frac{\partial^2 v}{\partial x^2}}\right) + \left(\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \boxed{v\frac{\partial^2 v}{\partial x \partial y}}\right) + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} + \boxed{w\frac{\partial^2 v}{\partial x \partial z}}\right) + \left(f\frac{\partial u}{\partial x}\right)$$
$$\boxed{-\left(\frac{\partial}{\partial y}\frac{\partial u}{\partial t}\right)} - \left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial x} + \boxed{u\frac{\partial^2 u}{\partial x \partial y}}\right) - \left(\frac{\partial v}{\partial y}\frac{\partial u}{\partial y} + \boxed{v\frac{\partial^2 u}{\partial y^2}}\right) - \left(\frac{\partial w}{\partial y}\frac{\partial u}{\partial z} + \boxed{w\frac{\partial^2 u}{\partial y \partial z}}\right) + \left(f\frac{\partial v}{\partial y} + v\frac{\partial f}{\partial y}\right)$$
$$= \boxed{-\frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y}} + \frac{1}{\rho^2}\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x} + \boxed{\frac{1}{\rho}\frac{\partial^2 p}{\partial x \partial y}} - \frac{1}{\rho^2}\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}$$

We can eliminate the second derivative pressure terms on the RHS (red boxes above). We will also combine the Eulerian u/v tendency terms (and switch the order of differentiation, green boxes) and combine the u, v, and w terms (purple) to get:

$$\boxed{\left(\frac{\partial}{\partial t}\frac{\partial v}{\partial x}-\frac{\partial}{\partial t}\frac{\partial u}{\partial y}\right)}+\boxed{u\left(\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 u}{\partial x\partial y}\right)+v\left(\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}\right)+w\left(\frac{\partial^2 v}{\partial x\partial z}-\frac{\partial^2 u}{\partial y\partial z}\right)}+$$
$$\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right)+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\boxed{\left(f\frac{\partial u}{\partial x}+f\frac{\partial v}{\partial y}\right)}+v\frac{\partial f}{\partial y}$$
$$=\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

Next, pull out the d/dt in the first term, d/dx in the second term, d/dy in the third term, and d/dz in the fourth term, and bring out the Coriolis parameter in the term in the second row (blue):

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)}+\boxed{u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)}+$$
$$\boxed{\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right)}+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\boxed{f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}+v\frac{\partial f}{\partial y}$$
$$=\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

For the first four terms, we will replace (dv/dx-du/dy) with zeta/relative vorticity, and rewrite the first term of the second row in terms of relative vorticity times divergence:

$$\boxed{\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}\right)=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}$$

To get to:

$$\boxed{\frac{\partial \zeta}{\partial t}}+\boxed{u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+w\frac{\partial \zeta}{\partial z}}+\boxed{\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\boxed{f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}+v\frac{\partial f}{\partial y}$$
$$=\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

We can define the Lagrangian derivative of Coriolis as:

$$\frac{df}{dt}=\cancel{\frac{\partial f}{\partial t}}+u\cancel{\frac{\partial f}{\partial x}}+v\frac{\partial f}{\partial y}+w\cancel{\frac{\partial f}{\partial z}}$$

But we can take advantage of the fact that the only non-zero term is the term that contains the meridional derivative and rewrite this as:

$$\frac{df}{dt} = v\frac{\partial f}{\partial y}$$

We combine the Eulerian and advective derivatives for relative vorticity (green and purple) into the total derviative, and replace the v(df/dy) term, we can condense the previous expression as:

$$\boxed{\frac{d\zeta}{dt}} + \boxed{\zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \boxed{f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} + \boxed{\frac{df}{dt}} = \frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

Combining the relative vorticity and Coriolis into a single total derivative and combining the terms that contain a multiplication of the divergence:

$$\boxed{\frac{d(\zeta + f)}{dt}} + \boxed{(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} + \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) = \frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

Finally, we move everything to the RHS to come up with the vorticity equation!

$$\frac{d(\zeta + f)}{dt} = \boxed{-(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)} - \boxed{\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)} + \boxed{\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x} - \frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)}$$

This is the vorticity equation in height coordinates (Martin 5.33). This equation states that the Lagrangian tendency (time rate of change following the flow) of the absolute vorticity consists of a divergence term (orange), tilting term (green), and solenoid term (yellow). See Martin Figures 5.11, 5.12, and 5.13 for schematic examples.

This same procedure can be repeated to derive a vorticity equation in isobaric (pressure) coordinates. Starting with:

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \omega\frac{\partial v}{\partial p} + fu = -\frac{\partial \phi}{\partial y}\right)$$

$$-\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial p} - fv = -\frac{\partial \phi}{\partial x}\right)$$

Which eventually leads to:

$$\frac{d(\zeta + f)}{dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \left(\frac{\partial \omega}{\partial x}\frac{\partial v}{\partial p} - \frac{\partial \omega}{\partial y}\frac{\partial u}{\partial p}\right)$$

In isobaric coordinates, we lose the solenoidal term. We can finally rewrite this as the isobaric vorticity equation:

$$\frac{\partial \zeta}{\partial t} = -\vec{V}\cdot\nabla(\zeta + f) - \omega\frac{\partial \zeta}{\partial p} - \boxed{(\zeta + f)(\nabla\cdot\vec{V})} + \boxed{\hat{k}\cdot\left(\frac{\partial \vec{V}}{\partial p}\times\nabla\omega\right)}$$