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- **In numerical weather prediction (NWP) the fully nonlinear primitive equations are solved by numerical means.**
- **In the atmosphere, the nonlinear advection process is a dominant factor.**
- **To get some idea of the methods used, we look at the simple problem of formulating time-integration algorithms for the solution of the simple advection equation.**

Recap. of Discretization Methods

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- **An *analytical* problem becomes an *algebraic* one.**
- **A problem with an *infinite* degree of freedom is replaced by one with a *finite* degree of freedom.**
- **A *continuous* problem goes over to a *discrete* one.**

The Finite Difference Method

We start by looking at the *Taylor expansion* of $f(x)$:

$$f(x + \Delta x) = f(x) + f'(x).\Delta x + \frac{1}{2}f''(x)\Delta x^2 + [O(\Delta x^3)] \qquad (1)$$

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We neglect these and obtain approximations for the derivative of $f(x)$ as follows:

$$f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} + O(\Delta x) = f'_F + O(\Delta x)$$

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Keeping only leading terms, we incur errors of order $O(\Delta x)$.

***We can do better than this*: subtracting (2) from (1) yields:**

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Adding (1) and (2) gives the corresponding expression for the second derivative:

$$f''(x) = \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{\Delta x^2} + O(\Delta x^2)$$

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Fourth-order accurate schemes are sometimes used in NWP, but *second order accuracy is more popular*.

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- **Demonstrate thus that the centered difference is of higher order accuracy.**

Grid Resolution and Accuracy

The size of the gridstep Δx determines the accuracy of the numerical scheme.

For the simple sine function the error depended on $k\Delta x = 2\pi\Delta x/L$, that is, on the ratio of the grid size Δx to the wave-length L.

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The higher the resolution, that is, the smaller the grid-size, the heavier the computational burden.

There is a *trade-off between resolution and accuracy*.

The Leapfrog Method

We consider the equation describing the conservation of a quantity $Y(x,t)$ following the 1D motion of a fluid flow:

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It is analogous to a factor of the wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} \right) Y = \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial t} - c \frac{\partial}{\partial x} \right) Y = 0 \, ,$$

and its general solution is $Y = Y(x - ct)$.

Since the advection equation is linear, we can construct a general solution from Fourier components

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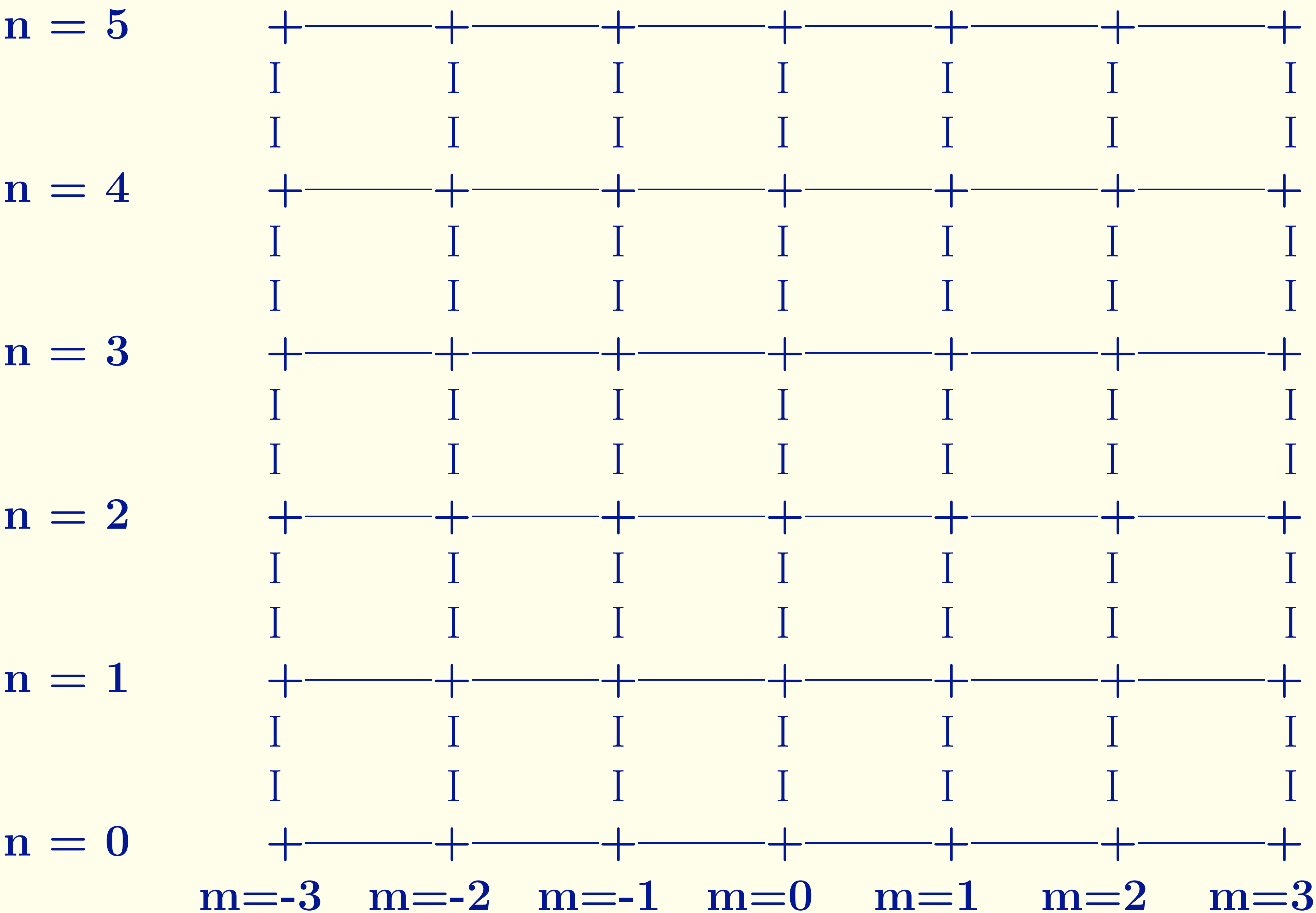
Let the variables x and t be represented by the horizontal and vertical axes. Positive time corresponds to the upper half plane. The initial data occur on the x-axis.

Space-Time Grid:

Space axis horizontal
Time axis vertical



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Then the (CTCS) finite difference approximation to the differential equation may be written as follows:

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Solving for the value at time $(n+1)\Delta t$ gives

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The ratio $\mu \equiv \dfrac{c\Delta t}{\Delta x}$ will be found to be crucial.

Inter-dependency of Points

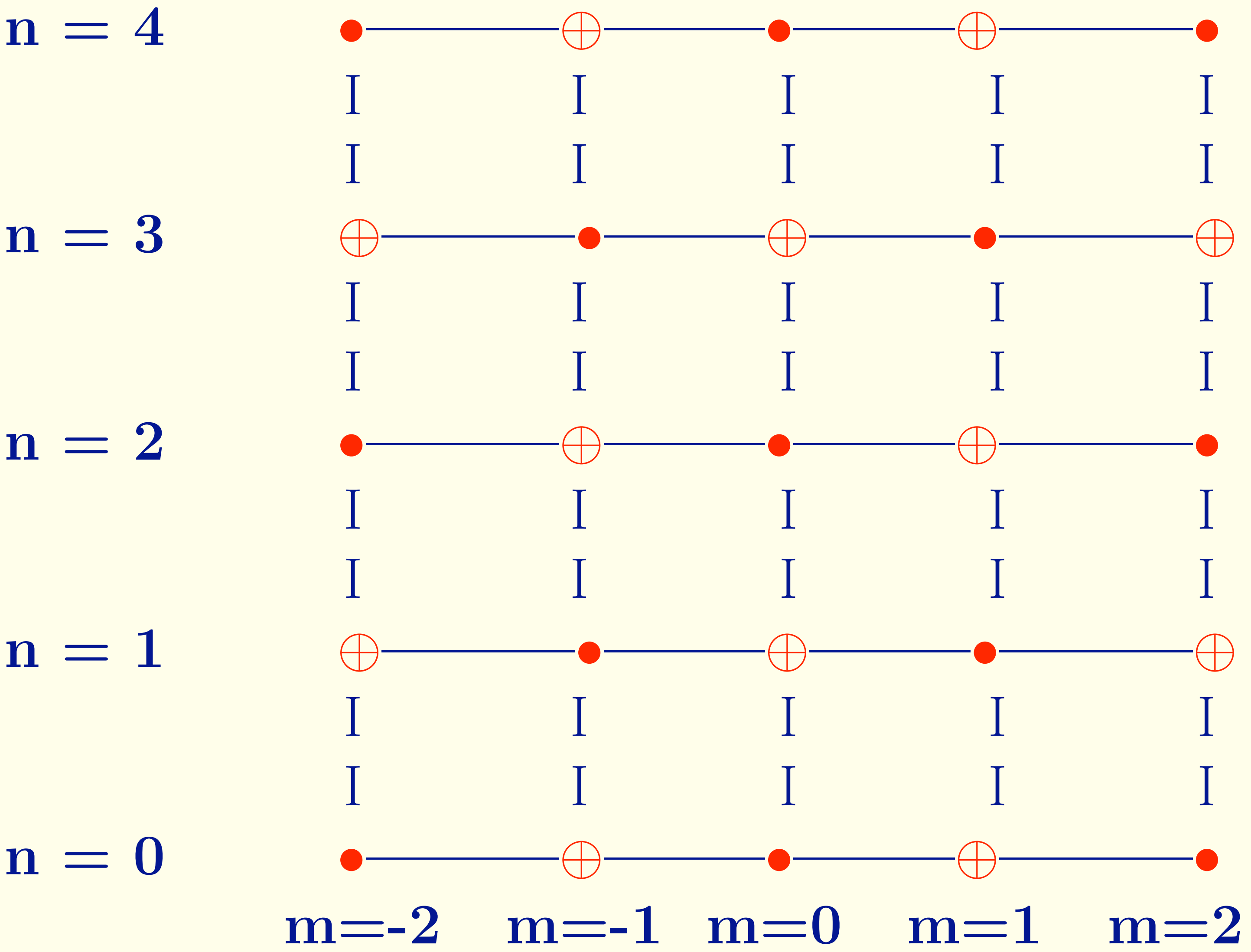
n + 1	$+$	$\oplus$	$+$
n	$\oplus$	$\bullet$	$\oplus$
n - 1	$+$	$\oplus$	$+$
	m-1	**m**	**m+1**

The evaluation of the equation at point $\bullet$ involves values of the variable at points $\oplus$. Solving for Y_m^{n+1} thus requires

$$Y_m^{n-1}, \qquad Y_{m-1}^{n} \qquad \textbf{and} \qquad Y_{m+1}^{n}.$$

The leapfrog scheme *splits the grid* into two independent sub-grids.

Grid Splitting



The finite difference grid splits into two sub-grids.

Steps must be taken to avoid divergence of the two solutions.

Recall the (CTCS) finite difference approximation:

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Then the whole procedure can be repeated to advance the solution to $(n+2)\Delta t$**, and so on.**

Question:

Under what conditions does the solution of the finite difference equation approximate that of the original differential equation?

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However, it turns out that this is not enough, and that the value of the ratio

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Substituting this in the equation, defining $\mu = c\Delta t/\Delta x$ **and dividing by a common factor gives**

$$A^2 + (2i\mu \sin k\Delta x)A - 1 = 0$$

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Substituting this in the equation, defining $\mu = c\Delta t/\Delta x$ **and dividing by a common factor gives**

$$A^2 + (2i\mu \sin k\Delta x)A - 1 = 0$$

This is a quadratic for the amplitude A**, with solutions**

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We write

$$A_\pm = -i\sigma \pm \sqrt{1-\sigma^2} \qquad \textbf{where} \qquad \sigma \equiv \mu \sin k\Delta x$$

We consider in turn the two cases.

Case I: $|\mu| \leq 1$

The quantity under the square-root sign is positive, so the modulus of A is given by

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The two values of the phase are

$$\psi_1 = -\arcsin\sigma$$

and

$$\psi_2 = \pi - \psi_1\,.$$

The solution of the equation may now be written

$$Y_m^n = \Big[D \exp(i\psi_1 n) + E \exp[i(-\psi_1 + \pi)n] \Big] \exp(ikm\Delta x)$$

where D **and** E **are arbitrary constants.**

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Exercise:

Check in detail the algebra leading to this solution.

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If the ratio μ is small, the physical mode solution is given approximately by

$$Y \approx a\exp[ik(m\Delta x - cn\Delta t)]$$

which is just the analytical solution.

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In this simple case, we can eliminate the computational mode. In general, it is much more difficult.

Case II: $|\mu| > 1$

Recall that the roots of the quadratic are

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This phenomenon is called computational instability.

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We thus require that $|\mu| \leq 1$. This condition for stability is known as the CFL Criterion:

$$\frac{c\Delta t}{\Delta x} \leq 1$$

after Courant, Friedichs and Lewy (1928), who first published the result.

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It implies that, <u>if we refine the space grid</u>, that is, decrease Δx, <u>we must also shorten the time step Δt</u>.

Thus, halving the grid size in a two dimensional domain results in an eightfold increase in computation time.

Unconditionally Stable Schemes.

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The semi-Lagrangian algorithm has enabled us to integrate the primitive equations using a time step of 15 minutes.

This can be compared to a typical timestep of 2.5 minutes for Eulerian schemes.

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We discuss semi-Lagrangian schemes in a later lecture.

End of §3.2.2