Chapter 2. The continuous equations

Intro to the equations used in NWP, filtering fast waves (remember Richardson's failure!)

2.1 Governing equations: Basically Newton's laws!

2.2 Atmospheric equations of motion on spherical cords.

2.3 Basic wave solutions in the atmosphere:
- Slow, weather waves (e.g., Rossby waves)
- Fast waves (gravity and sound waves)
- Their properties.
- Appendix: Equatorially trapped waves: (Kelvin and Rossby waves)

2.4 Filtering approximations: how to get rid of fast waves
- Quasi-geostrophic approximation
- Quasi-Boussinesq or anelastic approximation
- Hydrostatic approximation

2.5 Shallow water equations: Simple 2-D model to understand the full 3D-equations
- Terms that allow gravity waves (used in semi-implicit schemes to filter these waves)

2.6 Primitive equations and vertical coordinates
- General vertical coordinates
- Pressure coordinates
- Sigma and eta coordinates
- Isentropic coordinates.

2.1 Governing equations

V. Bjerknes (1904) manifesto: there are 7 equations with 7 unknowns that govern the evolution of the atmosphere:

- Newton's second law or conservation of momentum (3 equations for the 3 velocity components);

- the continuity equation or conservation of mass;

- the equation of state for ideal gases;

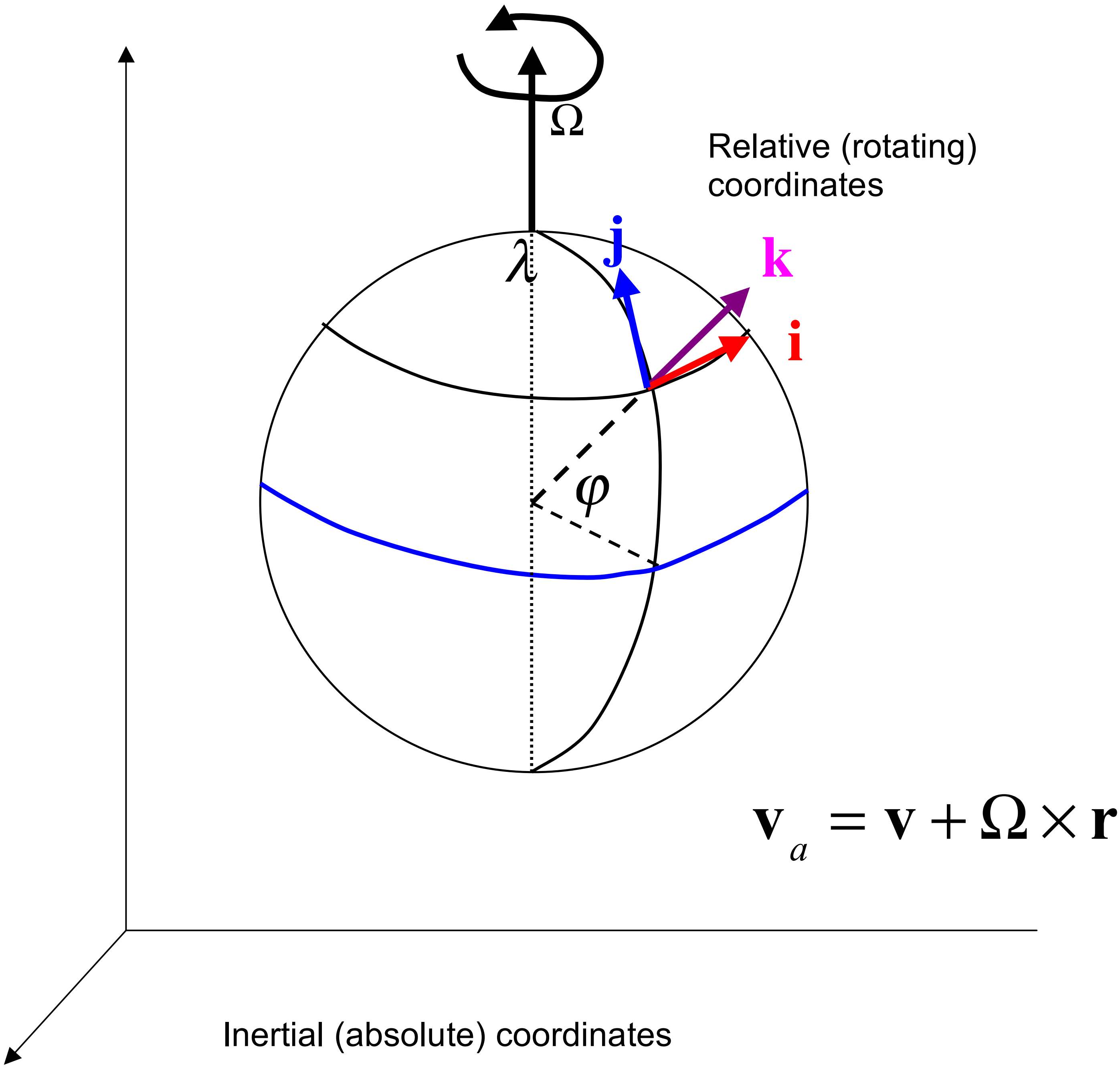
- the first law of thermodynamics or conservation of energy;

- a conservation equation for water mass.

So, we should be able to integrate them!

To these equations we have to

- add appropriate boundary conditions at the bottom and top of the atmosphere.


Ω
Relative (rotating) coordinates
j
k
i
λ
φ
v_a = v + Ω × r
Inertial (absolute) coordinates

Newton's second law or conservation of momentum:

On an inertial frame of reference, the absolute acceleration of a parcel of air in 3-dimensions is given by the physical forces per unit mass

$$\frac{d_a \mathbf{v}_a}{dt} = \mathbf{F}/m \qquad (1.1)$$

On a rotating frame of reference centered at the center of the earth:

absolute velocity $\mathbf{v}_a$, relative velocity $\mathbf{v}$, rotation with $\mathbf{\Omega}$, $\mathbf{r}$ is the position vector of the parcel:

$$\mathbf{v}_a = \mathbf{v} + \Omega \times \mathbf{r} \qquad (1.2)$$

More generally: the total time derivative of any vector on a rotating frame $\frac{d\mathbf{A}}{dt}$ is related to its total derivative in an inertial frame $\frac{d_a\mathbf{A}}{dt}$ by:

$$\frac{d_a \mathbf{A}}{dt} = \frac{d\mathbf{A}}{dt} + \Omega \times \mathbf{A} \qquad (1.3)$$

If we apply this formula to $\mathbf{A} = \mathbf{v}_a$,

$$\frac{d_a \mathbf{v}_a}{dt} = \frac{d\mathbf{v}_a}{dt} + \mathbf{\Omega} \times \mathbf{v}_a \qquad (1.4)$$

Substitute $\mathbf{v}_a = \mathbf{v} + \Omega \times \mathbf{r}$ into $\frac{d_a \mathbf{v}_a}{dt} = \frac{d\mathbf{v}_a}{dt} + \mathbf{\Omega} \times \mathbf{v}_a$:

$$\frac{d_a \mathbf{v}_a}{dt} = \frac{d\mathbf{v}}{dt} + 2\mathbf{\Omega} \times \mathbf{v} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \qquad (1.5)$$

On a rotating frame of reference there are two ***apparent*** forces per unit mass: the Coriolis force and the centrifugal force.

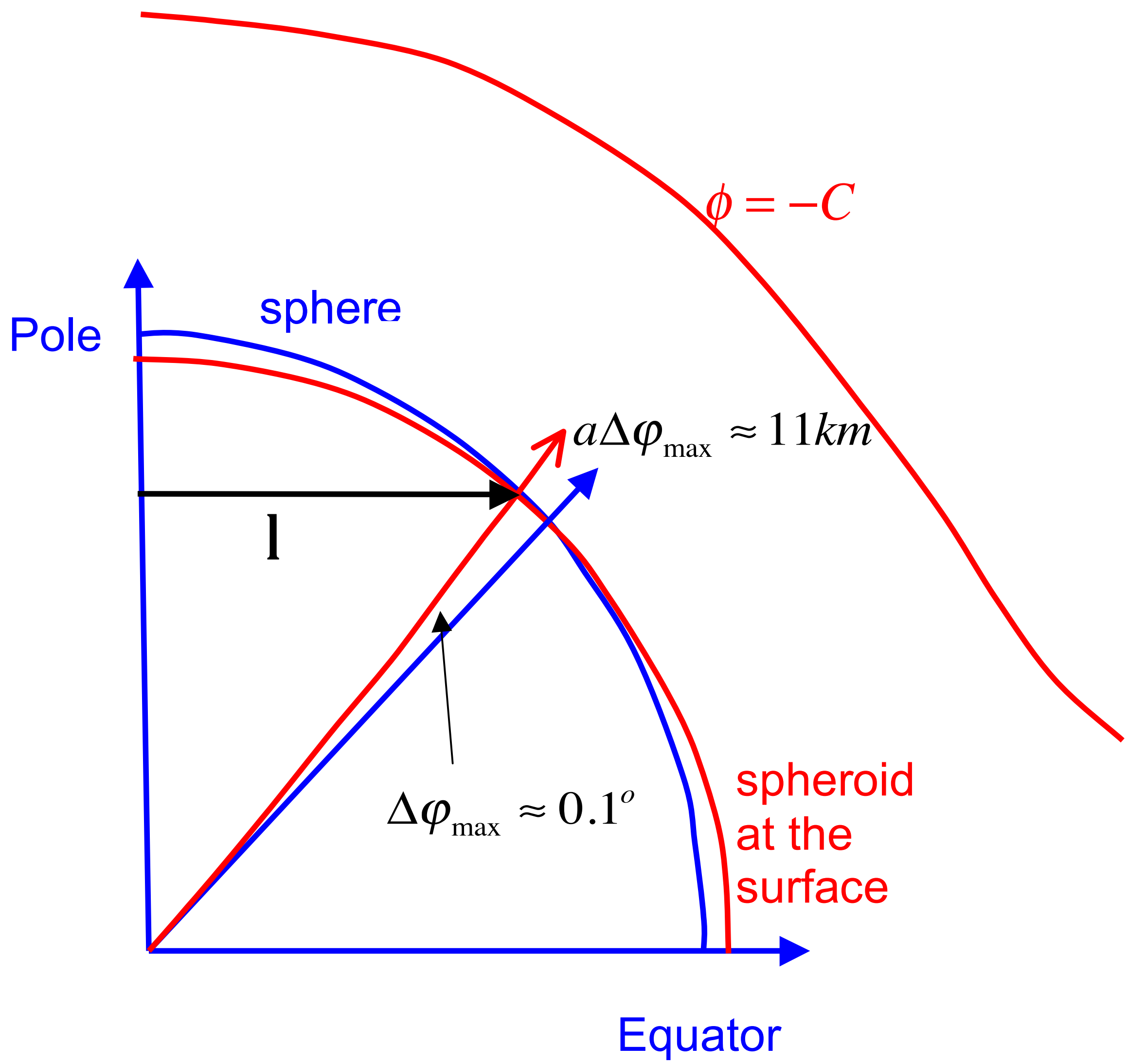
$\frac{d_a \mathbf{v}_a}{dt}$ represents the ***real*** forces acting on a parcel of air:

- pressure gradient force $\alpha \nabla p$,
- gravitational acceleration $\mathbf{g}_e = -\nabla \phi_e$
- frictional force $\mathbf{F}$.

So, in a rotating frame of reference moving with the Earth, the apparent acceleration is given by

$$\frac{d\mathbf{v}}{dt} = -\alpha \nabla p - \nabla \phi_e + \mathbf{F} - 2\mathbf{\Omega} \times \mathbf{v} - \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) \qquad (1.6)$$

Here $\alpha = 1/\rho$ is the specific volume (inverse of the density ρ), p is the pressure, ϕ_e is the Newtonian gravitational potential of the Earth. There is also the **tidal potential**, but its effects are negligible below about 100km.



Note: $\phi_e = -\frac{KM_e}{r}$;   at the surface $\phi_e = -\frac{KM_e}{r} = -\frac{KM_e}{a} = -g$

Far away $\phi_e \to 0$.

The total geopotential (with centrifugal effect) is

$$\phi = \phi_e - \frac{\Omega^2 l^2}{2} = -\frac{KM_e}{r} - \frac{\Omega^2 l^2}{2}$$

A surface of constant geopotential=-C is $r = \frac{KM_e}{C - \frac{\Omega^2 l^2}{2}}$

We can combine the centrifugal and the gravitational forces since they depend only on the position: this makes the centrifugal force "disappear"

$$-\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = \mathbf{\Omega}^2 \mathbf{l} = \nabla(\Omega^2 l^2 / 2)$$

$\mathbf{l}$ is the position vector from the axis of rotation to the parcel.

Define "geopotential" $\phi = \phi_e - \Omega^2 l^2 / 2$

so that the **apparent** gravity (including centrifugal acceleration) is given by

$$-\nabla\phi = \mathbf{g} = \mathbf{g_e} + \Omega^2 \mathbf{l} \qquad (1.7)$$

Define geographic latitude φ to be perpendicular to the geopotential ϕ.

(At the surface of the earth, the geographic latitude and the geocentric latitude differ by less than 10 minutes of a degree of latitude)

So, Newton's law on the rotating frame of the Earth (as we see it) is written as

$$\frac{d\mathbf{v}}{dt} = -\alpha\nabla p - \nabla\phi + \mathbf{F} - 2\mathbf{\Omega} \times \mathbf{v} \qquad (1.8)$$

Continuity equation or equation of conservation of mass:

Consider the mass of a parcel of air of density ρ

$$M = \rho \Delta x \Delta y \Delta z \qquad (1.9)$$

(assume an infinitesimal parcel, $\Delta x, \Delta y, \Delta z \to 0$)

If we follow the parcel with time, it conserves its mass.

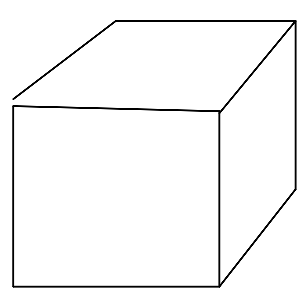
Total time derivative of a function $f(x, y, z, t)$
 (or substantial, individual or Lagrangian time derivative): time derivative following the parcel

$$\frac{df}{dt} = \left[\frac{\partial f}{\partial t}\right]_{x,y,z} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} =$$

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}u + \frac{\partial f}{\partial y}v + \frac{\partial f}{\partial z}w$$

Since a parcel conserves dry air mass: the total derivative of mass following parcel is zero

$$\frac{dM}{dt} = \frac{d\left(\rho \Delta x \Delta y \Delta z\right)}{dt} = 0$$



Take a logarithmic derivative (divide by the mass)

$$\frac{1}{M}\frac{dM}{dt} = \frac{1}{\rho}\frac{d\rho}{dt} + \frac{1}{\Delta x}\frac{d\Delta x}{dt} + \frac{1}{\Delta y}\frac{d\Delta y}{dt} + \frac{1}{\Delta z}\frac{d\Delta z}{dt} = 0$$

Now $\frac{1}{\Delta x}\frac{d\Delta x}{dt} = \frac{\partial u}{\partial x}$ as $\Delta x \to 0$

so that the continuity equation becomes

$$\frac{1}{\rho}\frac{d\rho}{dt} + \nabla_3.\mathbf{v} = 0 \qquad (1.10)$$

Again, the total derivative of any function $f(x,y,z,t)$, following a parcel, can be expanded as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}.\nabla f \qquad (1.11)$$

The total (or Lagrangian or individual) time derivative of a property is given by the local (or partial or Eulerian) time derivative (at a fixed point) plus the changes due to advection.

Expand $\frac{d\rho}{dt}$ in (1.10) and get an alternative form of the continuity equation, usually referred to as "in flux form":

$$\frac{\partial \rho}{\partial t} = -\nabla.(\rho \mathbf{v}) \qquad (1.12)$$

Equation of state for perfect gases:

The atmosphere can be assumed to be a perfect gas, for which the pressure p, specific volume α (or its inverse ρ, density) and temperature T are related by

$$p\alpha = RT \qquad (1.13)$$

where R is the gas constant for air. This equation indicates that given any two thermodynamic variables, the others are determined.

Thermodynamic energy equation or conservation of energy equation:

It expresses that if heat is applied to a parcel at a rate of Q per unit mass, this heat can be used to increase the internal energy $C_v T$ and/or to produce work of expansion:

$$Q = C_v \frac{dT}{dt} + p \frac{d\alpha}{dt} \qquad (1.14)$$

Coefficient of specific heat at constant volume C_v
Coefficient of specific heat at constant pressure C_p

Related by $C_p = C_v + R$.

With the equation of state $p\alpha = RT$ we derive a more commonly used form of the thermodynamic equation:

$$Q = C_p \frac{dT}{dt} - \alpha \frac{dp}{dt} \qquad (1.15)$$

Specific entropy S (not clear what it is!)

But we do know that the rate of change of *S* is

$\frac{ds}{dt} = \frac{Q}{T}$, diabatic heating divided by the absolute temperature.

Define potential temperature

$\theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{C_p}}$, p_0 is a reference pressure (1000hPa).

With this definition and (1.13, 1.15), it is easy to show (do it!) that the potential temperature and the specific entropy are related by

$$\frac{ds}{dt} = \frac{Q}{T} = C_p \frac{1}{\theta}\frac{d\theta}{dt} \qquad (1.16)$$

This shows that *potential temperature is individually conserved* in the absence of diabatic heating (isentropic or adiabatic flow).

When a parcel moves up, it expands, and cools adiabatically: *T* decreases. But the potential temperature (temperature the parcel would have at 1000hPa) remains constant:

If $Q = 0$ (adiabatic flow) then the potential temperature is individually conserved: $\frac{d\theta}{dt} = 0$ (= isentropic flow)

Equation of conservation of water vapor specific humidity (~mixing ratio) q, mass of water vapor/mass of dry air.

Finally, we include an **equation for conservation of water vapor mixing ratio q**. It simply indicates that the total amount of water vapor in a parcel is conserved as the parcel moves around, except when there are *sources* (evaporation E), and *sinks* (condensation C):

$$\frac{dq}{dt} = E - C \tag{1.17}$$

Conservation equation for other variable atmospheric constituents (e.g., CO2, O3) can be similarly written in terms of their corresponding sources and sinks.

Multiply $\frac{dq}{dt} = \frac{\partial q}{\partial t} + \mathbf{v}.\nabla q$ by ρ

multiply the continuity equation $\frac{\partial \rho}{\partial t} = -\nabla.(\rho \mathbf{v})$ by q

and add them: This gives the

Conservation of water in a "flux form":

$$\frac{\partial \rho q}{\partial t} = -\nabla.(\rho \mathbf{v} q) + \rho(E - C) \tag{1.18}$$

The flux form of the time derivative is very useful in the construction of models. The first term of the rhs of (1.18) is the convergence of the flux of q. Note that we can include additional similar conservation equations for additional tracers such as liquid water, ozone, etc, as long as we also include their corresponding sources and sinks.

Repeat the governing equations, which (without friction $\mathbf{F}$) are sometimes referred to as "*the Euler equations*":

$$\frac{d\mathbf{v}}{dt} = -\alpha \nabla p - \nabla \phi + \mathbf{F} - 2\mathbf{\Omega} \times \mathbf{v} \qquad (1.19)$$

$$\frac{\partial \rho}{\partial t} = -\nabla.(\rho \mathbf{v}) \qquad (1.20)$$

$$p\alpha = RT \qquad (1.21)$$

$$Q = C_p \frac{dT}{dt} - \alpha \frac{dp}{dt} \qquad (1.22)$$

$$\frac{\partial \rho q}{\partial t} = -\nabla.(\rho \mathbf{v} q) + \rho(E - C) \qquad (1.23)$$

W. Bjerknes (1904): "Seven equations with seven unknowns: $\mathbf{v} = (u, v, w), T, p, \rho$ or α, and q . With proper boundary conditions (e.g., at the surface and top of the atmosphere, we *should* be able to integrate them and forecast the weather!"

The fact that we succeeded in doing so is one of the most remarkable scientific achievements of the last 100 years.