Compact-Reconstruction WENO Methods for Compressible Flows

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Project Goal and Background

- Solve the Euler equations in two space dimensions.

Conservation of:

- Mass
- $x$-momentum
- $y$-momentum
- Energy
Background – Euler Equations

• Can also be written as:

\[
\frac{\partial q}{\partial t} + \frac{\partial F_i}{\partial x_i} = 0
\]

\[
q = \begin{bmatrix}
\rho \\
\rho u_1 \\
\rho u_2 \\
e
\end{bmatrix}, \quad F_1 = \begin{bmatrix}
\rho u_1 \\
\rho u_1 u_2 + p \\
\rho u_1 u_2 \\
(e + p)u_1
\end{bmatrix}, \quad F_2 = \begin{bmatrix}
\rho u_2 \\
\rho u_1 u_2 \\
\rho u_2^2 + p \\
(e + p)u_2
\end{bmatrix}
\]

State vector  Flux vectors
Background – Finite Volume Method

\[ \frac{\partial q}{\partial t} + \frac{\partial F_i}{\partial x_i} = 0 \]

Average over a cell

\[ \frac{1}{|\Omega_k|} \frac{d}{dt} \int_{\Omega_k} q \, dx = - \frac{1}{|\Omega_k|} \int_{\Omega_k} \frac{\partial F_i}{\partial x_i} \, dx \]

Divergence theorem

\[ \frac{d}{dt} \bar{q}_k = - \frac{1}{|\Omega_k|} \int_{\partial \Omega_k} F_i n_i \, ds \]
\[ \bar{q}_k = \frac{1}{|\Omega_k|} \int_{\Omega_k} q \, dx \]
Background – Finite Volume Method

\[
\frac{d}{dt} \bar{q}_k = -\frac{1}{|\Omega_k|} \int_{\partial \Omega_k} F_i n_i \, ds
\]

Known average over interior Unknown quantity on boundary

• How to reconstruct point values on the boundary from averages over the interior?
  – Many ways (polynomial fit, MUSCL, ...)
  – Want to avoid spurious oscillations near shocks
  – Want to resolve fine features efficiently
  – A Compact-Reconstruction Weighted Essentially Non-Oscillatory method ([1]) will be used in this project
Background – CRWENO

\[ \hat{u}_{k+1/2} \]

\[ \text{u}_k \text{ are known} \]

\[ \text{Find } \hat{u}_{k+1/2} \]

Optimal weights

\[ \frac{1}{5} \left( \frac{2}{3} \hat{u}_{j-1/2} + \frac{1}{3} \hat{u}_{j+1/2} = \frac{1}{6} \left( u_{j-1} + 5u_j \right) + O(h^3) \right) \]

\[ + \frac{1}{2} \left( \frac{1}{3} \hat{u}_{j-1/2} + \frac{2}{3} \hat{u}_{j+1/2} = \frac{1}{6} \left( 5u_j + u_{j+1} \right) + O(h^3) \right) \]

\[ + \frac{3}{10} \left( \frac{2}{3} \hat{u}_{j+1/2} + \frac{1}{3} \hat{u}_{j+3/2} = \frac{1}{6} \left( u_j + 5u_{j+1} \right) + O(h^3) \right) \]

\[ = \frac{3}{10} \hat{u}_{j-1/2} + \frac{3}{5} \hat{u}_{j+1/2} + \frac{1}{10} \hat{u}_{j+3/2} = \frac{1}{30} u_{j-1} + \frac{19}{30} u_j + \frac{1}{3} u_{j+1} + O(h^5) \]

Solutions to either side of a shock may be coupled
Background – CRWENO

Find $\hat{u}_{k+1/2}$

\[
\omega_1 \left( \frac{2}{3} \hat{u}_{j-1/2} + \frac{1}{3} \hat{u}_{j+1/2} = \frac{1}{6} \left( u_{j-1} + 5u_j \right) + O(h^3) \right) + \\
\omega_2 \left( \frac{1}{3} \hat{u}_{j-1/2} + \frac{2}{3} \hat{u}_{j+1/2} = \frac{1}{6} \left( 5u_j + u_{j+1} \right) + O(h^3) \right) + \\
\omega_3 \left( \frac{2}{3} \hat{u}_{j+1/2} + \frac{1}{3} \hat{u}_{j+3/2} = \frac{1}{6} \left( u_j + 5u_{j+1} \right) + O(h^3) \right)
\]

= Compact-Reconstruction Weighted Essentially Non-Oscillatory (CRWENO) scheme [1]
Background – CRWENO

• $\omega_i$ are solution dependent weights that:
  – Are $\approx 0$ if the corresponding stencil has a discontinuity
  – Revert to the optimal values (1/5, 1/2, 3/10) when the solution is smooth in all stencils

• Accomplished using smoothness indicators [3]:

$$\beta_i = \sum_{l=1}^{r-1} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} h^{2l-1} \left( p_i^{(l)}(x) \right)^2 dx$$
Background – CRWENO

\[ \omega_i = \frac{\alpha_i}{\sum_{i=1}^{3} \alpha_i} \]

\[ \alpha_i = \frac{\omega_{i, opt}}{(\epsilon + \beta_i)^p} \]

\[ \omega_{1, opt} = \frac{1}{5}, \omega_{2, opt} = \frac{1}{2}, \omega_{3, opt} = \frac{3}{10} \]

- \( \epsilon \) is a small \(<10^{-6}\) constant to avoid division by zero
- \( p \) is chosen to promote large denominators. Typically \( p = 2 \)
System becomes two independent systems

Example from [1]
Extension to Non-uniform Grids

\[
\frac{L + 1}{2L + 1} \hat{u}_{j-\frac{1}{2}} + \frac{L}{2L + 1} \hat{u}_{j+\frac{1}{2}} = \frac{1}{(2L + 1)(L + 1)} \bar{u}_{j-1} + \frac{L(2L + 3)}{(2L + 1)(L + 1)} \bar{u}_j
\]

\[
\frac{R}{2R + 1} \hat{u}_{j-\frac{1}{2}} + \frac{R + 1}{2R + 1} \hat{u}_{j+\frac{1}{2}} = \frac{R(2R + 3)}{(2R + 1)(R + 1)} \bar{u}_j + \frac{1}{(2R + 1)(R + 1)} \bar{u}_{j+1}
\]

\[
\frac{R + 1}{R + 2} \hat{u}_{j+\frac{1}{2}} + \frac{1}{R + 2} \hat{u}_{j+\frac{3}{2}} = \frac{R^2}{(R + 2)(R + 1)} \bar{u}_j + \frac{3R + 2}{(R + 2)(R + 1)} \bar{u}_{j+1}
\]

Optimal weights

\[
\omega_i = \frac{p_i}{p_1 + p_2 + p_3} \quad i = 1, 2, 3
\]

\[
p_1 = R^2(2L + 1)(R + 1)^2
\]

\[
p_2 = LR(2R + 1)(L + 1)(L + 2R + 2)
\]

\[
p_3 = L(L + 1)(R + 2)(L + R + 1)
\]

Center cell width
Left stretch ratio
Right stretch ratio

\[
h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}
\]

\[
L_j = \frac{x_{j-\frac{1}{2}} - x_{j-\frac{3}{2}}}{h_j}
\]

\[
R_j = \frac{x_{j+\frac{3}{2}} - x_{j+\frac{1}{2}}}{h_j}
\]
Approach

- Assume a Cartesian grid, possibly with obstacles and unequal spacing

- Grids expected to be \( \approx 100 \times 400 \) for test cases

- Allows 1-D reconstructions to be performed along lines of cells.
Approach

• Along each line of cells:
  – Use CRWENO to compute left- and right-biased approximations to state vectors at faces
  – Upwind the face fluxes (Roe flux difference splitting)

• For each cell:
  – Calculate the net flux through all faces (i.e., dq/dt)

• Advance in time by a 3rd-order, total-variation diminishing Runge-Kutta method

• Parallelizable
Characteristic Decomposition

- Hyperbolic systems have solutions comprised of traveling waves
- But the advected quantities may not be the variables of the state vector
- Possibility of strong shocks requires reconstruction of characteristic rather than conserved variables
  - Prevents oscillations and respects physics
Characteristic Decomposition

\[ \frac{\partial q}{\partial t} + \frac{\partial F(q)}{\partial x} = 0 \]

\[ \frac{\partial q}{\partial t} + J(q) \frac{\partial q}{\partial x} = 0 \]

\[ \frac{\partial q}{\partial t} + X \Lambda X^{-1} \frac{\partial q}{\partial x} = 0 \]

\[ X^{-1} \frac{\partial q}{\partial t} + \Lambda X^{-1} \frac{\partial q}{\partial x} = 0 \]

\[ \frac{\partial \alpha}{\partial t} + \Lambda \frac{\partial \alpha}{\partial x} = 0 \]

\[ \alpha = X^{-1} q \]

Decoupled system of scalar advection equations

\[ \frac{\partial}{\partial t} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ 0 \\ \lambda_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \]
Test Cases for Validation

Steady 1-D shock
• Converge to steady state and properly balance fluxes across a discontinuity

Isentropic vortex convection [1]
• Should preserve vortex strength and shape
• Verify accuracy for smooth problem with an exact solution
Test Cases for Validation

Mach 10 shock reflection [1]
• Resolve discontinuities not aligned with the grid

Mach 3 flow into step [2]
• Verify ability to handle obstacles
Software and Hardware Choices

- C++
- OpenMP for parallelism
- Eigen for linear algebra
- Intended to run on a laptop computer:
  - Intel Core T6500 processor (2 cores)
  - 2.10 GHz
  - 4 GB RAM
<table>
<thead>
<tr>
<th>Phase</th>
<th>Dates</th>
<th>Milestone</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Dates</td>
<td><strong>Proposal Presentation</strong></td>
</tr>
<tr>
<td>1</td>
<td>10/2</td>
<td>Validate the non-uniform CRWENO scheme. Select data structures, data formats, and boundary condition treatments.</td>
</tr>
<tr>
<td>2</td>
<td>10/3 - 10/31</td>
<td>Complete code to populate the grid data structure from grid specifications and initial conditions.</td>
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<tr>
<td>2</td>
<td>11/1 - 11/21</td>
<td>Complete code to perform the characteristic decompositions and spatial reconstructions.</td>
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<td>2</td>
<td>11/22 - 12/19</td>
<td>Complete code for time advancement.</td>
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<tr>
<td>3</td>
<td>12/20 - 1/16</td>
<td>Complete validation for the 1-D shock case.</td>
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<tr>
<td>3</td>
<td>1/17 - 1/30</td>
<td>Complete validation for isentropic vortex convection.</td>
</tr>
<tr>
<td>3</td>
<td>1/31 - 2/13</td>
<td>Complete validation for the Mach 10 shock reflection case.</td>
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<td>3</td>
<td>2/14 - 2/27</td>
<td>Complete validation for the Mach 3 flow with step case.</td>
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<td>4</td>
<td>3/14 - 4/1</td>
<td>Buffer time to either complete development and validation or possibly extend the code to handle curvilinear meshes.</td>
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<tr>
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<td>After 4/1</td>
<td>Finish code documentation and otherwise wrap up the project.</td>
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Deliverables

• Project proposal and this proposal presentation
• Validation of the non-uniform CRWENO scheme
• Documented code for the flow solver
• Results of the validation against test cases
• Mid-year and final written reports
• Mid-year and final oral presentations
References

