Project IV. Ensemble Kalman Filter

- Dates
  - 12min presentation: 2016.04.12 (& 04.14) in-class
    - Order to be announced
  - Report: 2016.04.15 5pm by email

- Objectives: Implementation of Ensemble Kalman Filter based on
  - Data Assimilation Framework developed in Project 1
  - Ensemble representation of minimum variance approach initially developed for OI in Project 2
  - Ensemble representation of EKF in Project 3
    - Understanding of dynamic propagation of uncertainty using tangent linear model, and accuracy
    - Schemes to improve the performance: Inflation of the error covariance
  - Enhancement of diagnostic tools
  - Additional scheme to support performance: localization of error covariance

Project V. Ensemble Kalman Filter

- Project Description (as reference – you are welcome to improvise)
  - Objective:
    - Implementation of at least
      - Perturbed-Observation EnKF
      - Serial Ensemble square-root KF (EnSRF)
      - [Local] Ensemble Transform KF ([L]ETKF)
    - Verification of ensemble analysis
      - in comparison with “adjusted” EKF analysis, given the choice of the EnKF method (slide 3)
  - Diagnostics: Having successfully implemented the EnKF,
    - Comparison of the results with:
      - EKF (& 3D-Var/OI) and 4D-Var
      - truth: actual error $|x^b-x^t|^2$ & $|x^a-x^t|^2$ vs estimated error by $\text{var}(x^b_m)$ & $\text{var}(x^a_m)$
    - Study the effects of
      - Size of ensemble (start from $M=N$, and in/decrease)
      - Covariance inflation
      - Covariance localization
    - Case study (not statistics) in which EnKF failed, and identify the cause, for example
      - Observations: Insufficient obs? Too large obs error?
      - Background: Failed forecast due to nonlinearity?
      - Analysis: Inappropriate inflation and Localization?
Basic Ensemble Operations (Notation)

- In \( \mathbf{x} \)-space based on ensemble \( \{ \mathbf{x}_m \} \)
  
  - Ensemble \( \mathbf{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_M) \in \mathbb{R}^{N \times M} \)
  
  - Mean \( \bar{\mathbf{x}} = \frac{1}{M} \sum_{m=1}^{M} \mathbf{x}_m \in \mathbb{R}^N \)
  
  - Spread \( \mathbf{P} = \frac{1}{M-1} \mathbf{XX}^T \in \mathbb{R}^{N \times N} \)

- In \( \mathbf{y} \)-space based on ensemble \( \{ \mathbf{y}_m \} \)
  
  - Ensemble \( \mathbf{Y} = (\mathbf{y}_1, \ldots, \mathbf{y}_M) = (h(\mathbf{x}_1), \ldots, h(\mathbf{x}_M)) \in \mathbb{R}^{L \times M} \)
  
  - Mean \( \bar{\mathbf{y}} = \frac{1}{M} \sum_{m=1}^{M} \mathbf{h}(\mathbf{x}_m) \in \mathbb{R}^L \)
  
  - Spread \( \mathbf{Y} = (\mathbf{y}_1 - \bar{\mathbf{y}}, \ldots, \mathbf{y}_M - \bar{\mathbf{y}}) \in \mathbb{R}^{L \times M} \)
  
  - Projection of covariances (\( \approx \) holds if linear, i.e., \( h(\mathbf{x}) = \mathbf{H} \mathbf{x} \))
    
    \( \mathbf{P} \mathbf{H}^T = \frac{1}{M-1} \mathbf{YX}^T \in \mathbb{R}^{L \times \ell} \) & \( \mathbf{H} \mathbf{P} \mathbf{H}^T = \frac{1}{M-1} \mathbf{YY}^T \in \mathbb{R}^{\ell \times \ell} \)

- Essential schemes for analysis
  
  - Inflation
  
  - Localization
Formulation: Perturbed observation EnKF & ETKF

- Perturbed observation EnKF
  \[ \mathbf{x}_m^+ = \mathbf{x}_m^- + \mathbf{K}(\mathbf{y}_m^- - \mathbf{y}_m^0) \]
  \[ \mathbf{K} = \frac{1}{M-1} \hat{\mathbf{X}}^\top (\mathbf{1}_M - \frac{1}{M-1} \hat{\mathbf{Y}}^\top \mathbf{R}^{-1}) \in \mathbb{R}^{N \times N} \]
  \[ \mathbf{y}_m^0 = \mathbf{y}^0 + \mathbf{e}_m^0 \quad \mathbf{e}_m^0 \text{ is drawn from } \mathcal{N}(\mathbf{0},\mathbf{R}^0) \]

- ETKF
  \[ \mathbf{\hat{x}}^+ = \mathbf{\hat{x}}^- + \mathbf{\hat{X}}^- \mathbf{\omega}^w \quad \mathbf{\omega} \in \mathbb{R}^M \]
  \[ \mathbf{\hat{x}}^- = \mathbf{\hat{x}}^- \mathbf{\omega}^- \quad \mathbf{\omega} \in \mathbb{R}^{M \times M} \]

  where \( (\hat{\mathbf{P}}^w)^{-1} = (M-1) \]
  \[ \hat{\mathbf{P}} = (\mathbf{P}^w)^{-1} + (\hat{\mathbf{Y}}^\top (\mathbf{R}^-)^{-1} \hat{\mathbf{Y}})^{-1} \]
  and
  \[ \mathbf{\omega}^w = \hat{\mathbf{P}} (\hat{\mathbf{Y}}^\top (\mathbf{R}^-)^{-1} \mathbf{y}^o - \mathbf{y}^b) \]
  \[ \mathbf{W}^w = (M-1) \hat{\mathbf{P}}^{w/2} \]

Formulation: EnSRF

- EnSRF by serial assimilation
  
  - Starting from
    
    \[ \mathbf{\hat{x}}^{(0)} = \mathbf{\hat{x}}^0 \]
    \[ \mathbf{\hat{X}}^{(0)} = \mathbf{\hat{X}}^0 \]

  - For \( l=1,\ldots,L \):
    
    \[ \mathbf{\hat{x}}^{(l)} = \mathbf{\hat{x}}^{(l-1)} + \mathbf{\hat{X}}^{(l-1)} (\mathbf{y}_m^o - h_\alpha (\mathbf{\hat{x}}^{(l-1)})) \]
    \[ \mathbf{\hat{X}}^{(l)} = \mathbf{\hat{X}}^{(l-1)} - \beta (\mathbf{\hat{X}}^{(l-1)} \mathbf{\hat{X}}^{(l-1)} ) \]
    [ \( = \{1-\beta (\mathbf{\hat{X}}^{(l)}) \mathbf{\hat{X}}^{(l)} \} \mathbf{\hat{X}}^{(l-1)} \) if \( h_\alpha (\mathbf{x}) \) is linear] 

  where
  \[ \mathbf{\hat{P}}^{(l-1)} = \frac{1}{M} \sum_{m=1}^{M} \mathbf{h}_m (\mathbf{\hat{x}}^{(l-1)} ) \]
  \[ \mathbf{\hat{Y}}^{(l-1)} = (\mathbf{h}_m (\mathbf{\hat{x}}^{(l-1)} ) - \mathbf{\hat{P}}^{(l-1)} ) \in \mathbb{R}^{N \times M} \]
  \[ (\mathbf{\mathbf{y}}^{(l)}, \mathbf{\alpha}^{(l)}, \beta^{(l)}) = (\mathbf{\hat{Y}}^{(l)} (\mathbf{\hat{Y}}^{(l)} \mathbf{\hat{Y}}^{(l)} ) / (M-1), \mathbf{\hat{Y}}^{(l)} + \mathbf{R}^{(l)} , (1+\sqrt{\mathbf{R}^{(l)} / \alpha^{(l)} } )^{-1} ) \]
  \[ \mathbf{K}^{(l)} = (M-1)^{-1} \alpha^{-1} \mathbf{\hat{X}}^{(l-1)} (\mathbf{\hat{Y}}^{(l)} \mathbf{\hat{Y}}^{(l)} ) \in \mathbb{R}^{N \times N} \]

  - At \( L=1 \)
    
    \[ \mathbf{\hat{x}}^o = \mathbf{\hat{x}}^{(L)} \]
    \[ \mathbf{\hat{x}}^o = \mathbf{\hat{x}}^{(L)} \]
Essential Schemes: Inflation & Localization

- **Inflation**: common to all EnKF methods
  - For simplicity, multiplicative inflation to background ensemble spread:
    \[ \hat{X} \rightarrow \rho \hat{X} \quad \text{with} \quad \rho \geq 1 \]
  - Other approaches to inflation
    - Additive: \[ \hat{X} \rightarrow \hat{X} + \hat{X} \quad \text{where} \quad \hat{X}' \in \mathbb{R}^{N \times M} \text{ is small random matrix} \]
    - Relaxation to background:
      \[ \hat{X} \rightarrow \mu \hat{X}^* + (1-\mu) \hat{X}^* \quad \text{with} \quad \mu \in [0,1] \]

- **Localization**:
  - Model dependent (Lorenz 3 variable model does not require localization)
  - Typical localization function: Gaussian
    \[ c(r) = \begin{cases} \exp(-r^2 / R^2) & \text{if} \quad r \leq R, \; (\approx 3R) \\ 0 & \text{otherwise} \end{cases} \]

Essential Schemes: Localization Based on Observation

- For Perturbed Observation EnKF & EnSRF
  \[ K = \frac{1}{(M-1)} \hat{X} \hat{Y}^T \left( \frac{1}{(M-1)} \hat{Y} \hat{Y}^T + R \right)^{-1} \]

- Spurious correlation in \( \hat{X} \hat{Y}^T \) and \( \hat{Y} \hat{Y}^T \)

- Localization is imposed based on distance from observation
  \( \hat{X}_{(l)} \hat{Y}_{(j)}^T \rightarrow c(r_{(l)}) \cdot \hat{X}_{(l)} \hat{Y}_{(j)}^T \)
  \( \hat{Y}_{(j)} \hat{Y}_{(i)}^T \rightarrow c(r_{(j)}) \cdot \hat{Y}_{(j)} \hat{Y}_{(i)}^T \)

- \( r_{(l)} \): distance between
  - obs position \( \ell : Y_{(l)} \) [subset of \( Y \)]
  - grid point \( n : \hat{X}_{(l)} \) [subset of \( \hat{X} \)]
Essential Schemes: Localization Based on Observation

- For ETKF (becomes LETKF)
  \[ \hat{P}^* = (\hat{P}^\text{a})^{-1} + (\hat{Y}^\text{a})^T (R_o)^{-1} (\hat{Y}^\text{a})^{-1} \]

  - ETKF is performed at each grid point \(i\)
  - Inverse of observation error covariance \(R_o\) for observation \(i\) used are weighted based on distance from the grid point
    \[ (R_o)^{-1} \Rightarrow c(r_i)(R_o)^{-1} \]