2.9 Bernoulli's Theorem

Bernoulli's theorem is one simple but powerful result which follows from the equations derived in the last section. Start from Euler's equation, Equation 2.40, and assume steady, inviscid and incompressible flow. Euler's equation then reduces to

$$\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla\left(\frac{p}{\rho}\right) - \nabla\Phi$$

Use the vector identity

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + \mathbf{A}\times(\nabla\times\mathbf{B})$$

with $\mathbf{A} = \mathbf{B} = \mathbf{u}$ to rewrite the nonlinear advection term as follows:

$$\mathbf{u}\cdot\nabla\mathbf{u} = \nabla\left(\frac{\mathbf{u}\cdot\mathbf{u}}{2}\right) - \mathbf{u}\times(\nabla\times\mathbf{u}) \tag{2.45}$$

Use this identity and rearrange the Euler equation into the following form:

$$\mathbf{u}\times(\nabla\times\mathbf{u}) = \nabla\left(\frac{\mathbf{u}\cdot\mathbf{u}}{2} + \Phi + \frac{p}{\rho}\right) \tag{2.46}$$

Finally, take scalar product with $\mathbf{u}$ of this equation. Notice that the vector $\mathbf{u}\times(\nabla\times\mathbf{u})$ is perpendicular to $\mathbf{u}$, so its scalar product with $\mathbf{u}$ is 0. Therefore,

$$\mathbf{u}\cdot\nabla B = 0 \quad \text{where } B = \frac{\mathbf{u}\cdot\mathbf{u}}{2} + \Phi + \frac{p}{\rho} \tag{2.47}$$

The 'Bernoulli potential' B is conserved along a streamline. Conservation of B is essentially an energy relationship, relating the kinetic energy, potential energy and pressure energy along a streamline. The pressure term arises because we are dealing with a fluid parcel rather than a simple body. The theorem can be used to relate p, $\mathbf{u}\cdot\mathbf{u}$ and Φ along a streamline. For example, consider flow along a pipe of variable cross section. If the area of the pipe normal to the flow is $A(s)$, s being the distance along the streamline, and if the flow is steady, conservation of matter requires

$$A_0 u_0 = A(s)u(s)$$

Further, assume that the pipe is horizontal so Φ does not change and that the values of p and u at entry are p_0 and u_0, then

$$u_0^2 + \frac{p_0}{\rho} = \left(\frac{A_0}{A(s)}\right)^2 u_0^2 + \frac{p(s)}{\rho} \tag{2.48}$$

Thus, where the flow speed is larger, the pressure must be smaller and vice versa. Notice that Bernoulli's theorem can only be used to compare points on the same streamline. It cannot be used to compare the properties of points on different streamlines. Neither can it be used to compare the properties of turbulent flow at different points, for in that case, the flow is unsteady and streamlines are no longer trajectories.

If $A(s)$ becomes so small that

$$\left(\frac{A_0}{A(s)}\right)^2 u_0^2 > u_0^2 + \frac{p_0}{\rho}$$

then $p(s)$ becomes negative. This is clearly unphysical and indicates a breakdown of the assumptions in Bernoulli's theorem. In fact, this criterion marks the transition to 'cavitating flow', in which voids open up and collapse in the fluid so that it becomes highly unsteady. Of course, Bernoulli's theorem is not applicable beyond the transition to cavitating flow. A foaming mountain stream is a good natural example of a cavitating flow.

For compressible flow, such as characterizes the atmosphere, Equation 2.47 is incomplete. The reason for this is that if the density changes so that the volume of the parcel changes, the parcel does work against the pressure force exerted by its neighbours. To include this work, note that the first law of thermodynamics states that

$$\dot{q} = c_v \mathbf{u} \cdot \nabla T + p\mathbf{u} \cdot \nabla\left(\frac{1}{\rho}\right) \tag{2.49}$$

(see Equation 2.22) where $\dot{q}$ is the rate at which heat is added to the parcel. This equation states that the heating can be manifested either as a change of the internal energy of the parcel, or by the parcel doing work against its surroundings. As well as mechanical energy, account must be taken of the heat energy added to a parcel. So in place of Equation 2.46, we now have

$$\dot{q} = c_v \mathbf{u} \cdot \nabla T + p\mathbf{u} \cdot \nabla\left(\frac{1}{\rho}\right) + \frac{1}{\rho}\mathbf{u} \cdot \nabla p + \mathbf{u} \cdot \nabla\left(\frac{\mathbf{u} \cdot \mathbf{u}}{2} + \Phi\right) \tag{2.50}$$

This is now a total energy equation for the parcel, including the rates of change of both mechanical and internal energy. Assume zero heating, so the flow is adiabatic, and rearrange to give

$$\mathbf{u} \cdot \nabla\left(\frac{\mathbf{u} \cdot \mathbf{u}}{2} + \Phi + c_v T + \frac{p}{\rho}\right) = 0 \tag{2.51}$$

Now from the ideal gas equation, $p/\rho = RT$ and $c_v + R = c_p$, the specific heat at constant pressure, so that Equation 2.51 can finally be rewritten as:

$$\mathbf{u} \cdot \nabla B = 0 \quad \text{where } B = \mathbf{u} \cdot \mathbf{u}/2 + \Phi + c_p T \tag{2.52}$$

In words, Bernoulli's theorem for steady compressible flow is that the sum of kinetic energy, potential energy and 'enthalpy' for a parcel is conserved following the streamlines. Thus, if a parcel moves along a streamline to a place where the flow is accelerated (e.g., above an aerofoil), the temperature must fall to conserve B. Associated with this adiabatic cooling, the pressure must drop according to

$$Tp^{-\kappa} = \theta p_0^{-\kappa}$$

One may say that an aircraft flies because the air above the wing is colder than that below the wing, as a result of adiabatic expansion and compression. If the temperature change along a streamline is sufficiently small, Equation 2.52 reduces to the incompressible form, Equation 2.47. The condition for this to be the case is that the Mach number, the ratio of the typical flow speed to the sound speed, be small, a condition to which we shall return in Section 4.5. Strong winds in the lee of mountains are an example of Bernoulli's theorem acting in an atmospheric context. If a streamline descends from the top of a 1 km high mountain to near the surface, then Equation 2.52 suggests that winds as strong as $140\,\mathrm{m\,s^{-1}}$ are theoretically possible.

2.10 Heating and water vapour

Air parcels do not evolve exactly adiabatically, and so the heating and cooling terms on the right-hand side of the thermodynamic equation, Equation 2.35, can be significant. Air parcels both gain and lose heat by a variety of processes. Often, the rates of heating or cooling are sufficiently slow that the parcel motion can be treated as approximately adiabatic for periods shorter than 1 or 2 days. For example, clear air in the troposphere loses heat, eventually to space, by the emission of infrared thermal radiation. The typical cooling rate is around $1\text{–}2\,\mathrm{K\,day^{-1}}$, which is an order of magnitude smaller than the Eulerian rates of temperature change associated with a typical weather system. Similarly, the Earth–atmosphere system is heated by sunlight, most of which reaches the ground in clear conditions. A variety of processes then mix this heat through the depth of the atmosphere. A typical excess of shortwave heating over longwave cooling is around $10^2\,\mathrm{W\,m^{-2}}$. If this heat were mixed throughout the depth of the troposphere, it would result in a rate of temperature change of around $1\,\mathrm{K\,day^{-1}}$, comparable to the temperature change from longwave cooling. So on the basis of these calculations, one might conclude that for synoptic timescales of not more than a few days, the effects of heating and cooling will be no more than a small correction to the dynamics of air parcels.

However, this conclusion is of course misleading, and the reason it is misleading is because the discussion so far has neglected the role of water in the atmosphere. It is indeed remarkable that so much can be said about atmospheric dynamics without including the effects of water. After all, in the popular mind, the major application of meteorology is to predict rain and cloud events. Yet many of the major features of midlatitude weather systems can be elucidated without detailed discussion of the effects of moisture. But even if moisture does not change the character of basic dynamical processes, changes of phase of water substance have important effects on the details of the evolution of weather systems and on the vigour of certain processes.