3.6 The Taylor–Proudman theorem (or vanishing of thermal wind)

The impact of rapid rotation can be very dramatic. 'Rapid' rotation means a flow for which the Rossby number $Ro = U/fL$, introduced in Section 2.4, is small compared to 1. When the rotation is slow, so the Rossby number is large compared to 1, the effects of rotation are no more than a small correction and can often be ignored completely. So, in the context of flow in the Earth's midlatitude atmosphere, is the rotation of the Earth rapid or slow?

By way of example, in the Earth's midlatitude troposphere, a typical flow speed U is $10\,\mathrm{m\,s^{-1}}$ and a typical large length scale over which U varies is 10^6 m. The rotation rate of the Earth is (to the nearest power of 10) around $10^{-4}\,\mathrm{s^{-1}}$. The corresponding Rossby number is around 0.1. So rotation dominates. For ocean flows, which are generally much slower, the Rossby number may be as small as 10^{-3}.

One might say that appreciable, or indeed, rapid rotation is the defining characteristic of geophysical fluid dynamics. Stratification also plays an important role, but rotation is crucial. In this section, some dramatic results in simple situations will be described. Later sections will show how these extreme examples relate to the flows observed in more realistic circumstances.

Consider a tank of fluid which has been standing on a rotating turntable for sufficient time to come to rest relative to the rotating frame of reference fixed in the turntable (Figure 3.4). It is then gently stirred to generate some weak motion relative to the turntable. We shall make three assumptions about the flow:

1. The relative motions are weak, in the sense that the Rossby number $U/\Omega L$ is small compared to 1.
2. The fluid is incompressible, with constant density, so that $(1/\rho)\nabla p \rightarrow \nabla(p/\rho)$.

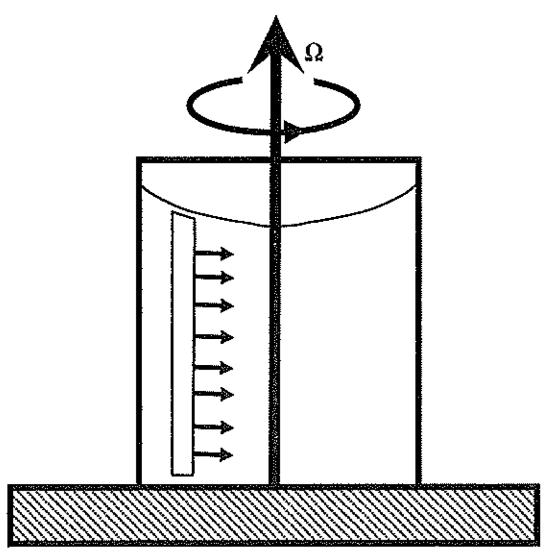

Figure 3.4 A tank of homogeneous fluid on a rotating turntable

3. The flow is inviscid in the sense that viscous stresses are much smaller than other forces acting.

The equations of motion for the tank of fluid are

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = -2\boldsymbol{\Omega}\times\mathbf{u} - \nabla\Phi_e - \frac{1}{\rho}\nabla p + \mathbf{F} \tag{3.18}$$

Incorporating the three assumptions given earlier provides

$$0 = -2\boldsymbol{\Omega}\times\mathbf{u} - \nabla\Phi_e - \nabla\left(\frac{p}{\rho}\right) \tag{3.19}$$

Take the curl of this equation; the gravitational and pressure gradient terms, being pure gradients, have zero curl, and so

$$\nabla\times(\boldsymbol{\Omega}\times\mathbf{u}) = 0 \tag{3.20}$$

From the standard vector identity,

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} - \mathbf{B}(\nabla\cdot\mathbf{A}) - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) \tag{3.21}$$

and noting that for incompressible fluid, the continuity equation is $\nabla\cdot\mathbf{u}=0$, the curl of the equation of motion reduces to

$$(\boldsymbol{\Omega}\cdot\nabla)\mathbf{u} = 0 \tag{3.22}$$

That is, the velocity vector **u** cannot vary in a direction parallel to the rotation axis. Let the rotation axis be parallel to the vertical (z-) axis, parallel to unit vector **k**. Then, splitting Equation 3.22 into its vertical and horizontal components gives

$$\begin{aligned} \Omega\frac{\partial w}{\partial z} &= 0 \quad \text{(a)} \\ \Omega\frac{\partial \mathbf{v}}{\partial z} &= 0 \quad \text{(b)} \end{aligned} \tag{3.23}$$

Now the boundary condition at the bottom of the tank is $w=0$. Equation 3.23(a) implies that if w is zero for any particular value of z, it must be zero at all other values of z. Thus, rapid rotation suppresses vertical motion.

The constraints on the horizontal components of the flow implied by Equation 3.23(b) are not quite as strong. No particular value is implied for **v**. However, whatever value **v** has at any point in the fluid, it must have the same value at all other points on a line parallel to the rotation axis which passes through that point. That

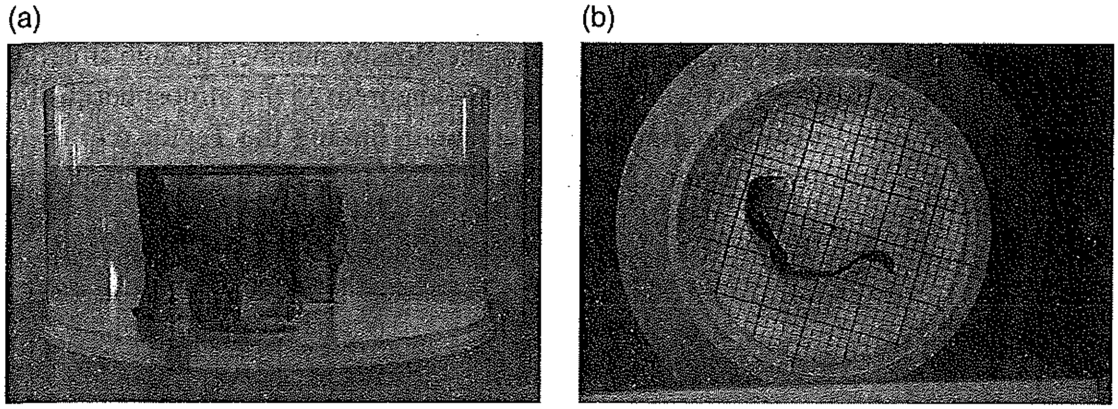

Figure 3.5 Taylor's inkwall experiment. (a) top view and (b) side view

means that the fluid must move as coherent columns orientated parallel to the rotation axis, with the same velocity at every level in the fluid.

The effect of rapid rotation on fluid flow expressed by the Taylor–Proudman theorem is both dramatic and counter-intuitive. It is as if the rotation imparts a degree of rigidity to the flow. Many beautiful experiments can be devised to illustrate the Taylor–Proudman theorem. One of the simplest is 'Taylor's inkwall experiment'. A tank of water on a turntable is allowed to spin up and is then lightly stirred. The flow is visualized by dropping a droplet of dense dye into the tank. As the dye falls, it leaves a vertical trail behind. After a short time, this trail is spread and distorted by the weak motions in the fluid. But the velocity field is the same at every level, according to the Taylor–Proudman theorem. So each column is pulled out into a curving thin sheet, with the same distortion at every level, a sheet which Taylor called an 'inkwall'. Seen from above, the round spot of dye left by the droplet is pulled and sheared into long curving streamers, but with a high degree of vertical coherence (Figure 3.5).

Another experiment which illustrates the Taylor–Proudman theorem consists of towing a shallow obstacle slowly across the base of a rotating tank. The tank is filled with water which has been allowed to spin up to rest in the rotating frame of reference of the tank. At a level below the summit of the shallow obstacle, the flow must part and move around the obstacle. But because the Taylor–Proudman theorem operates, the flow at every other level, even those well above the obstacle, must also pass around the obstacle edge. As a result, a column of fluid extends through the depth of the tank, from the obstacle to the surface, apparently attached to the obstacle. The rest of the flow passes around that column. It is as if the obstacle had been extended to fill the entire depth of the tank. Such a column is called a 'Taylor column'. There is some evidence for such structures in the ocean, where gentle currents pass over isolated seamounts on the oceans' abyssal plains. Taylor columns have been suggested as the origin of long-lived features in the outer fluid layers of the giant planets: Jupiter's 'Great Red Spot', in particular, has been interpreted in these terms.

The Taylor–Proudman theorem does not apply directly to the atmosphere, principally because the density is not constant. Also, the Rossby number, though small, is not infinitesimal. However, we can generalize the theorem to the atmospheric situation, and we can devise a continuum of behaviour linking the Taylor regime to more realistic situations. Chapter 12 and later chapters will address these issues.