Analysis Methods in Atmospheric and Oceanic Science

AOSC 652

Numerical Integration

Week 7, Day 3

- **Review HW #6**
- **General help with HW #7**

14 Oct 2016

AOSC 652: Analysis Methods in AOSC

Between now and Monday:

IDL Tract:

Please read Chapters 1, 2, and 3 of Bowman

An Introduction to Programming with IDL

Python Tract:

Please read Chapters 2, 3, 4, and 5 of DeCaria,

Python Programming and Visualization for Scientists

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Gaussian Integration

Function	2 Nodes	10 Nodes	Error, 10 Nodes
$3x^2$	1.000	1.000	-7.42×10^{-13}
$5x^4$	0.972	1.000	-8.18×10^{-13}
$9x^8$	0.674	1.000	-8.92×10^{-13}
$13x^{12}$	0.376	1.000	-8.29×10^{-13}

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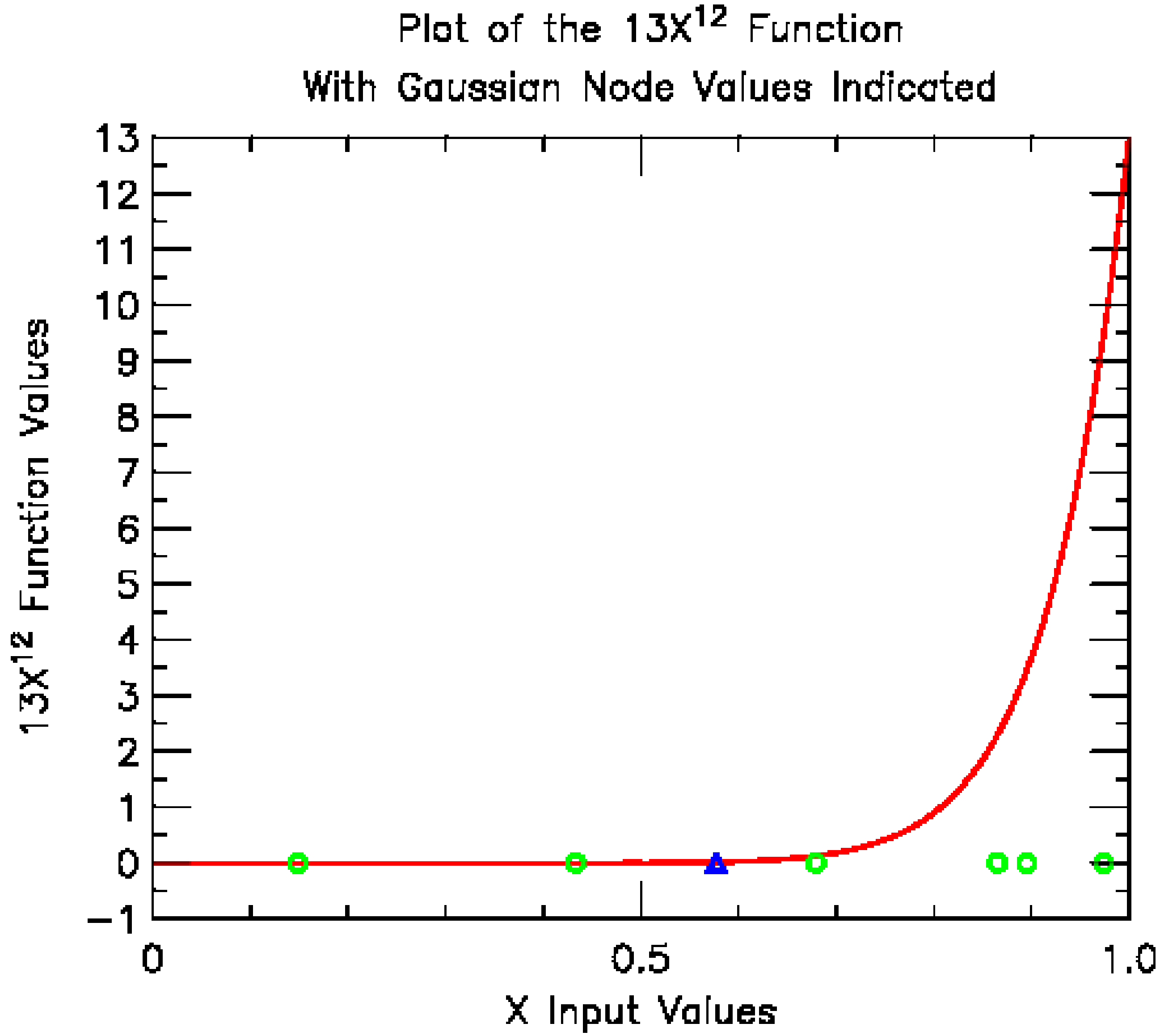
Gaussian Integration



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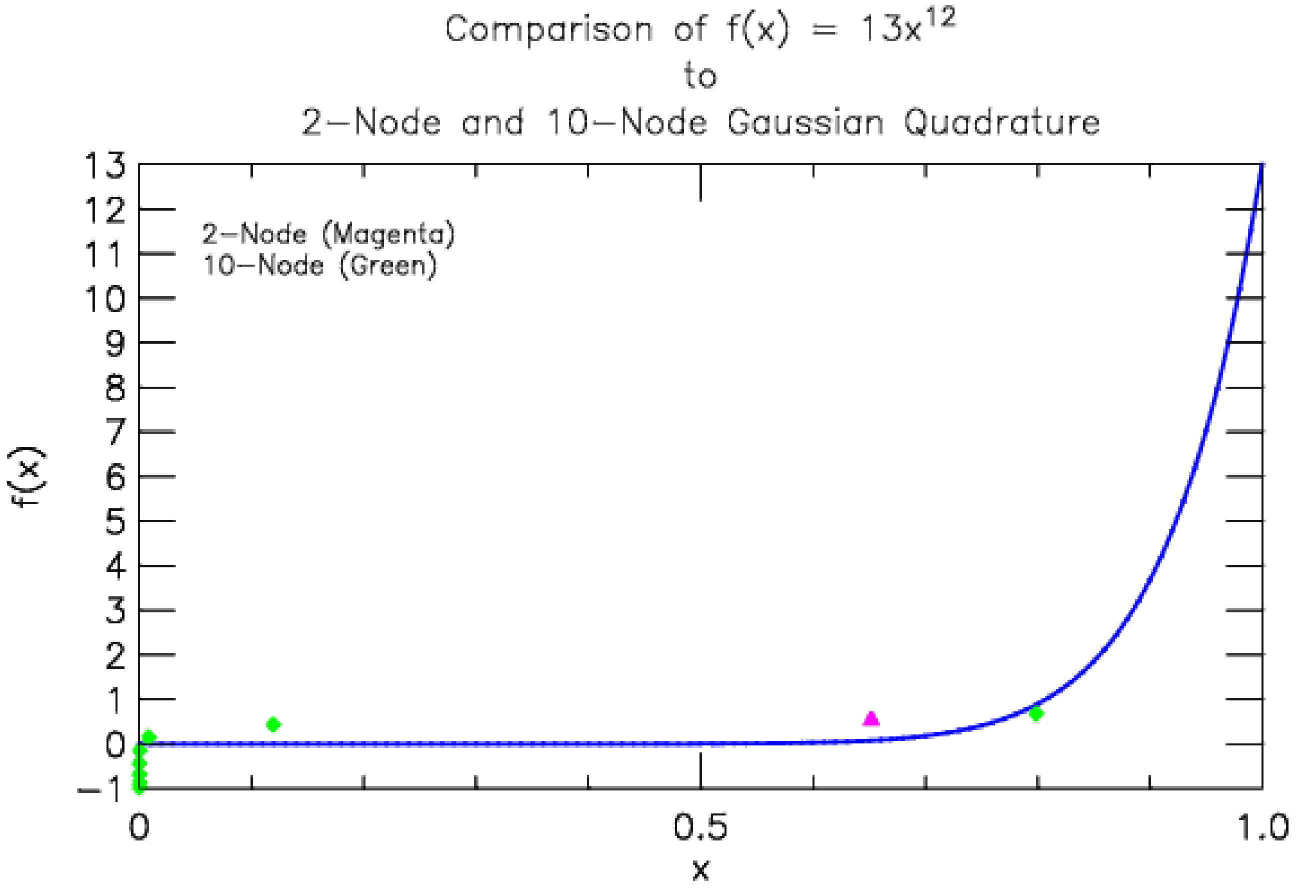
Gaussian Integration



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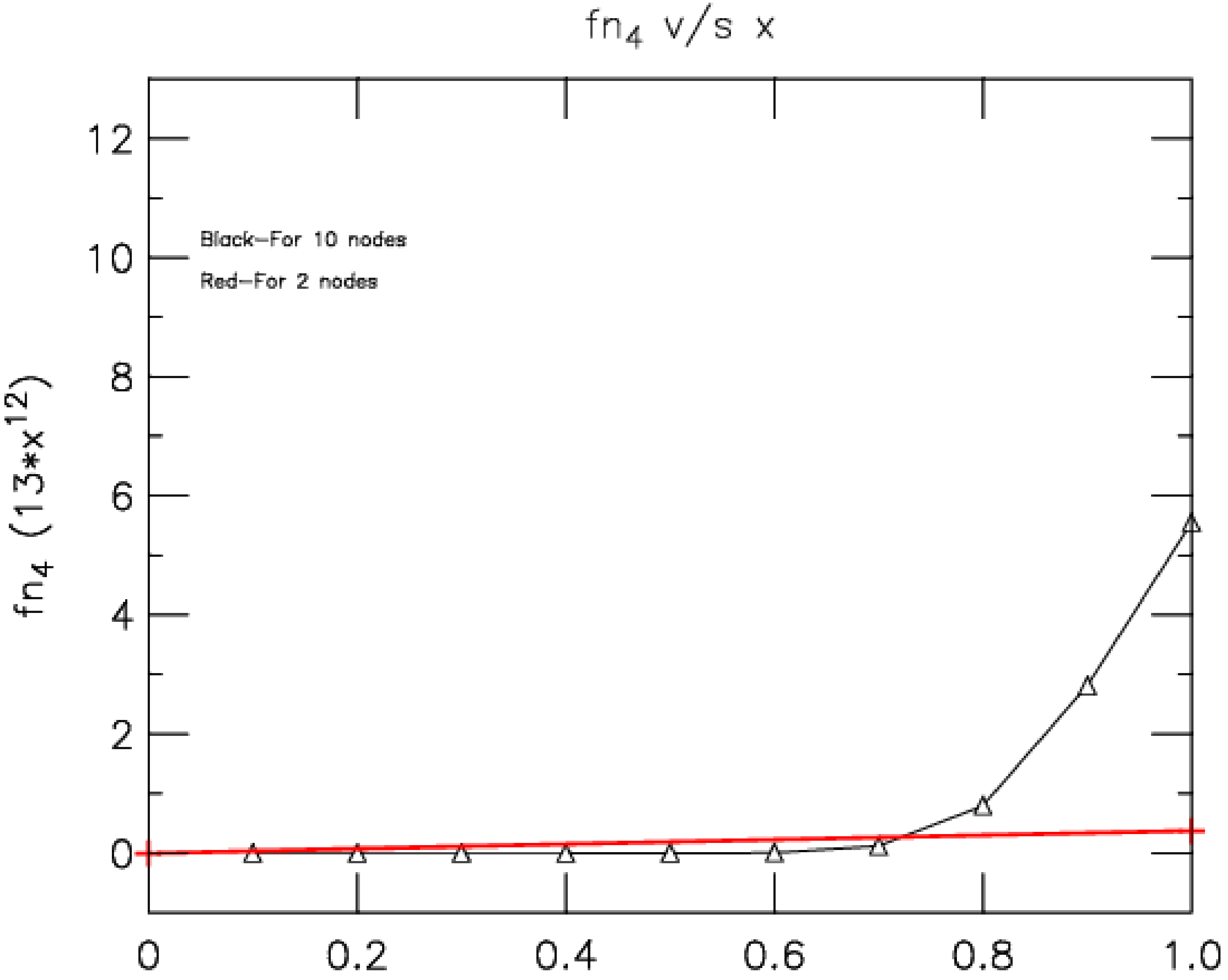

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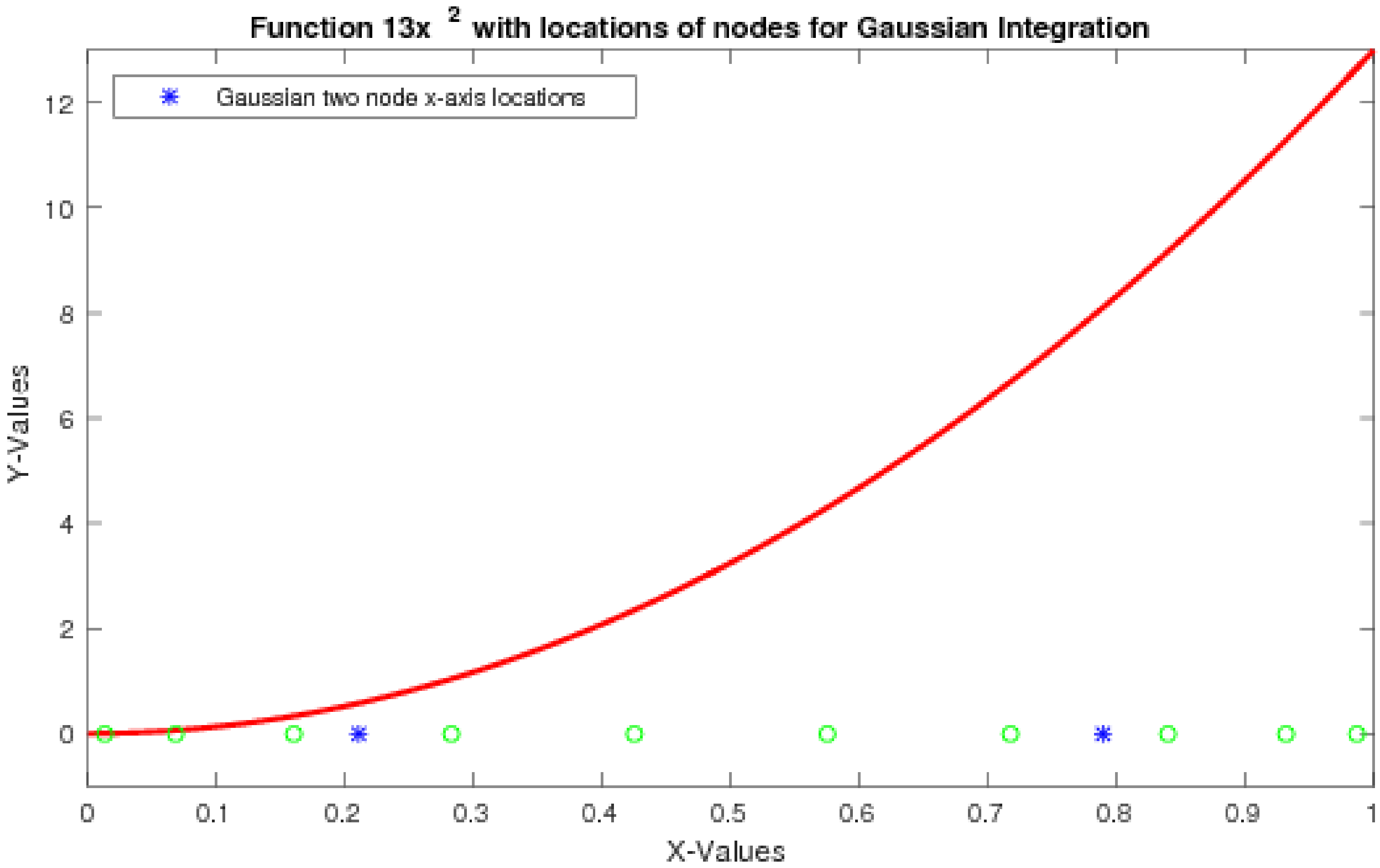

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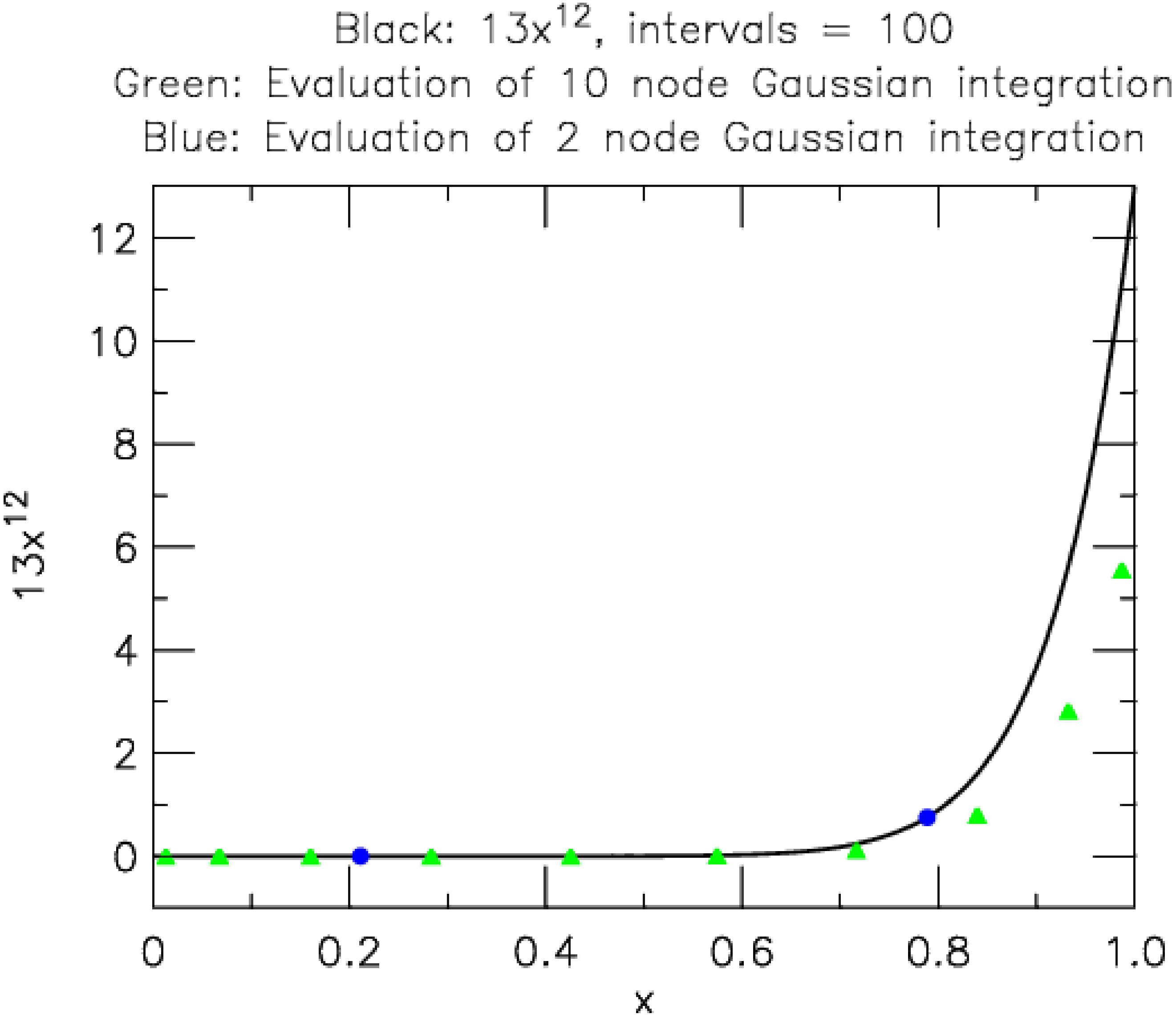
Gaussian Integration



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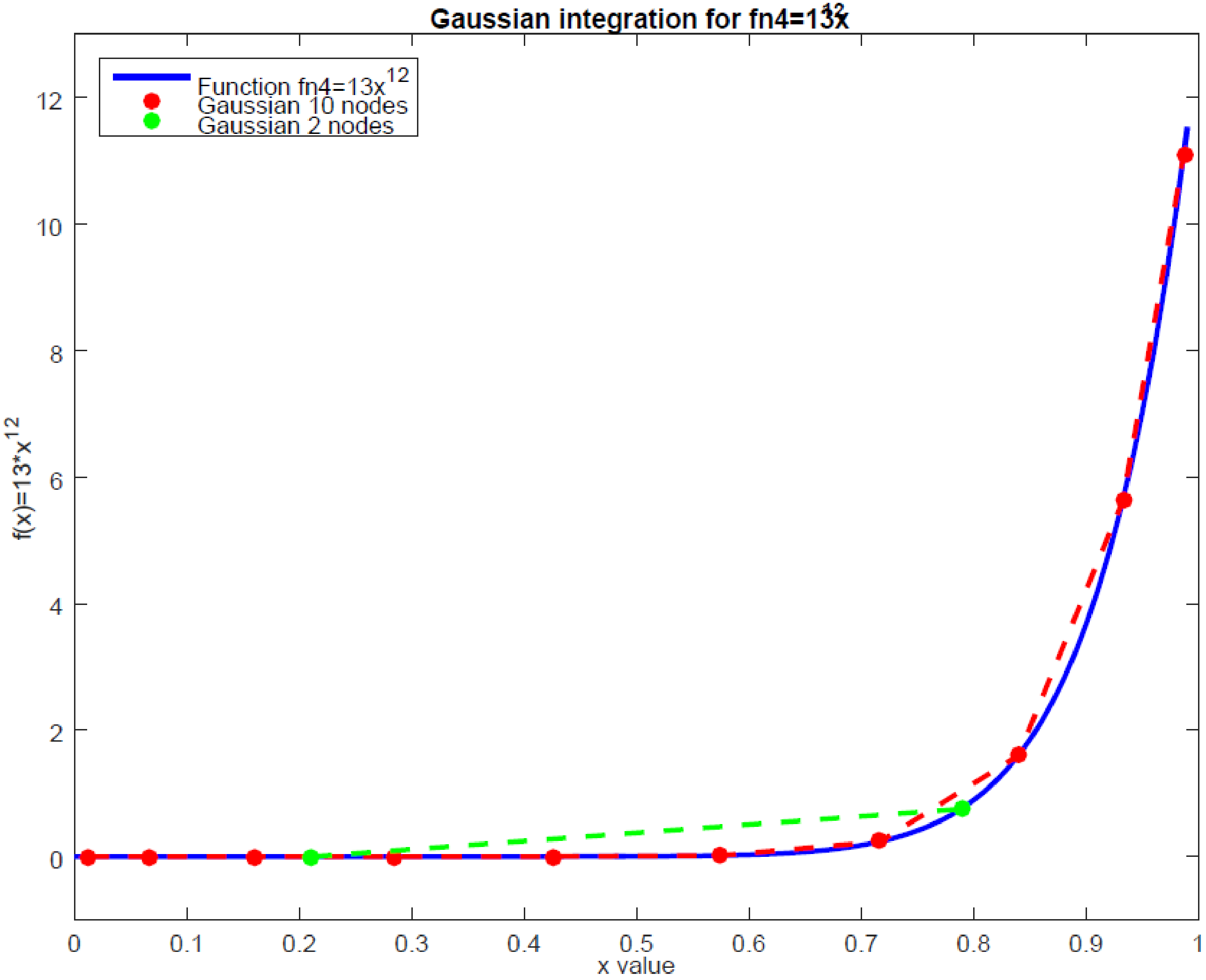
Gaussian Integration

Note: Gaussian quadrature acts as if we had fit a series of high order polynomials tp the function, so a plot showing the nodes connected by straight lines does not represent how this method works



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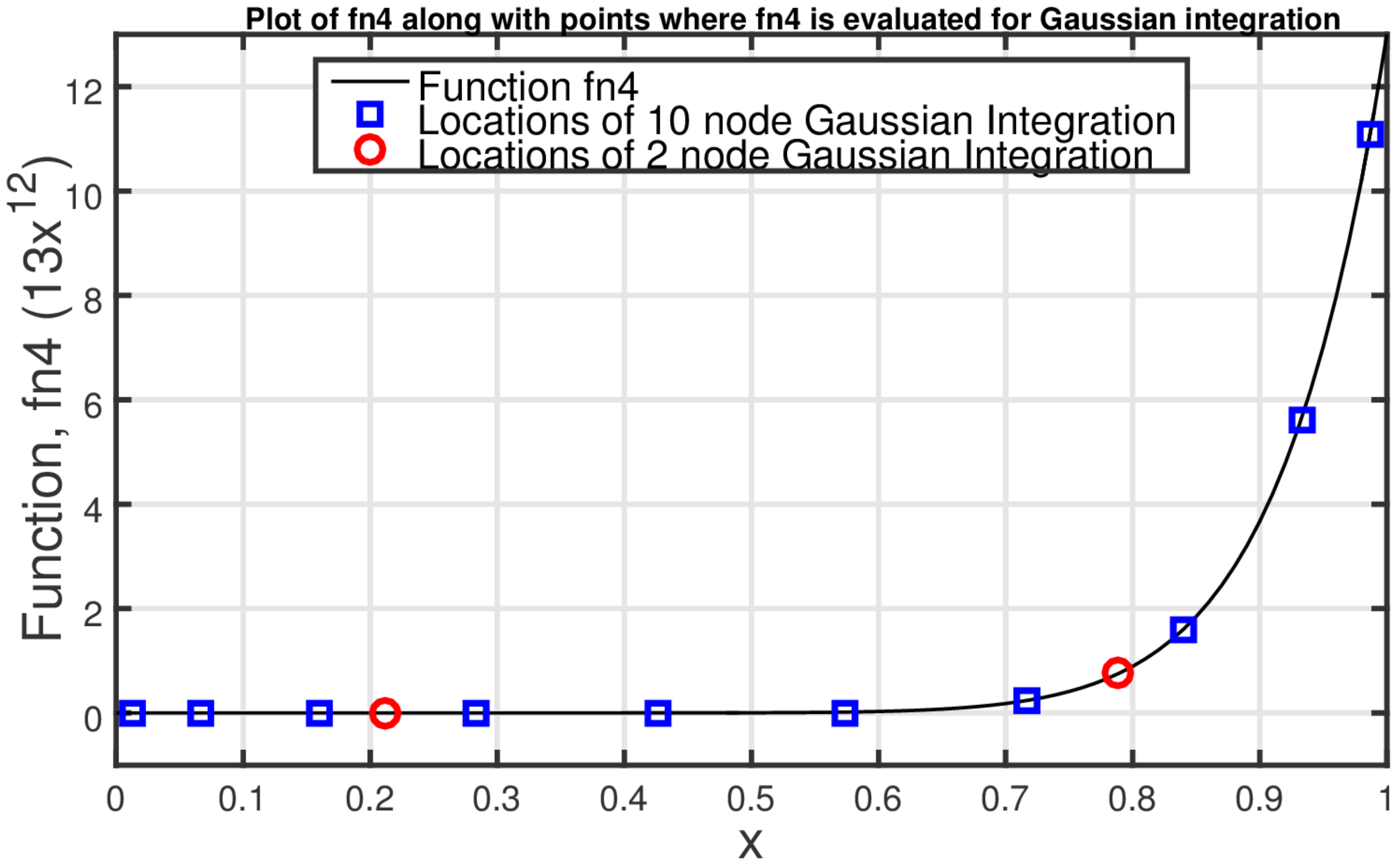
Gaussian Integration



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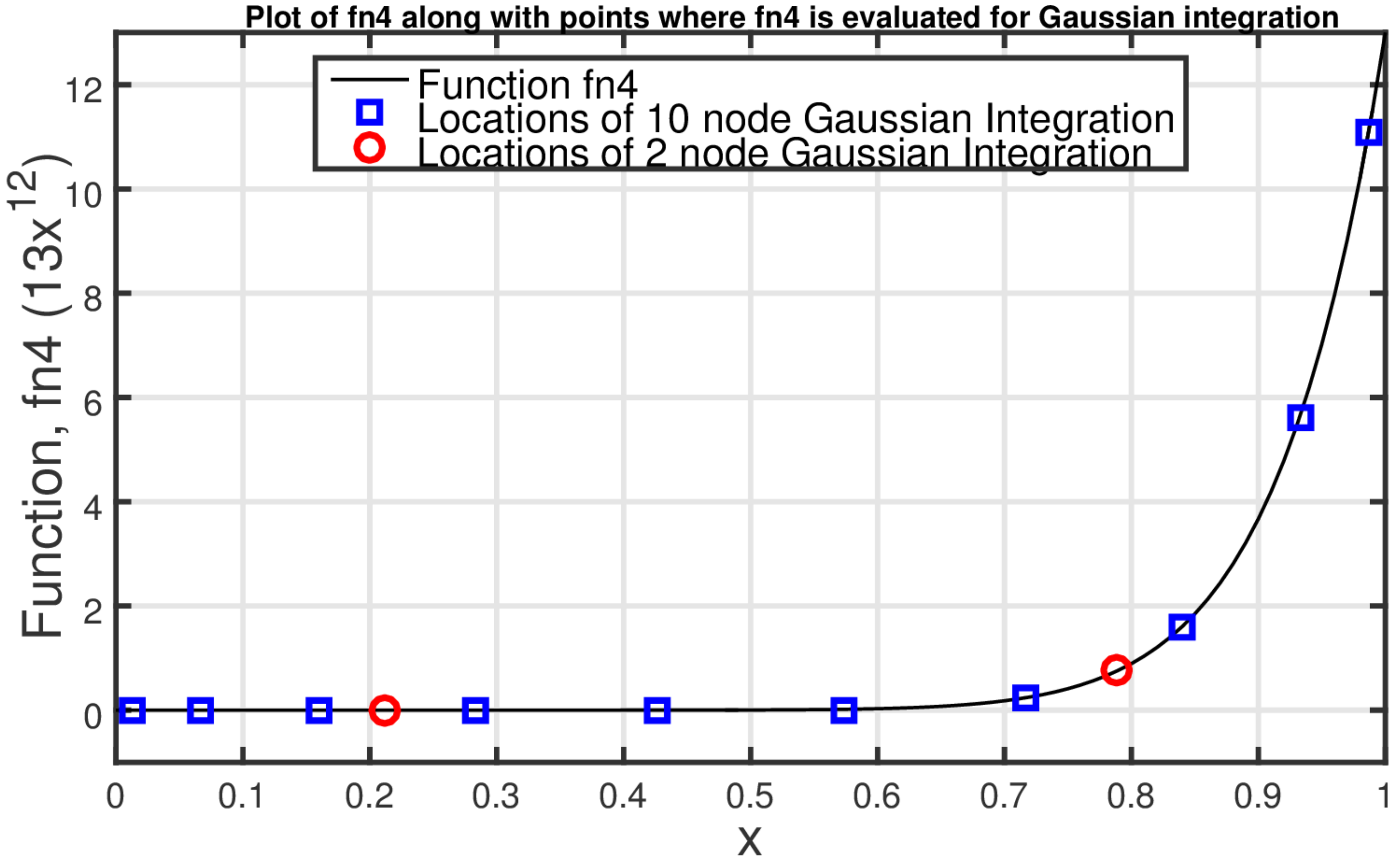
Gaussian Integration

Less than full credit for part if your answer dealt in general statements such as
“for 2 node approach, the assumed polynomials are only up to order 3,
Which is not good enough” or “2 nodes doesn’t represent the whole plot”



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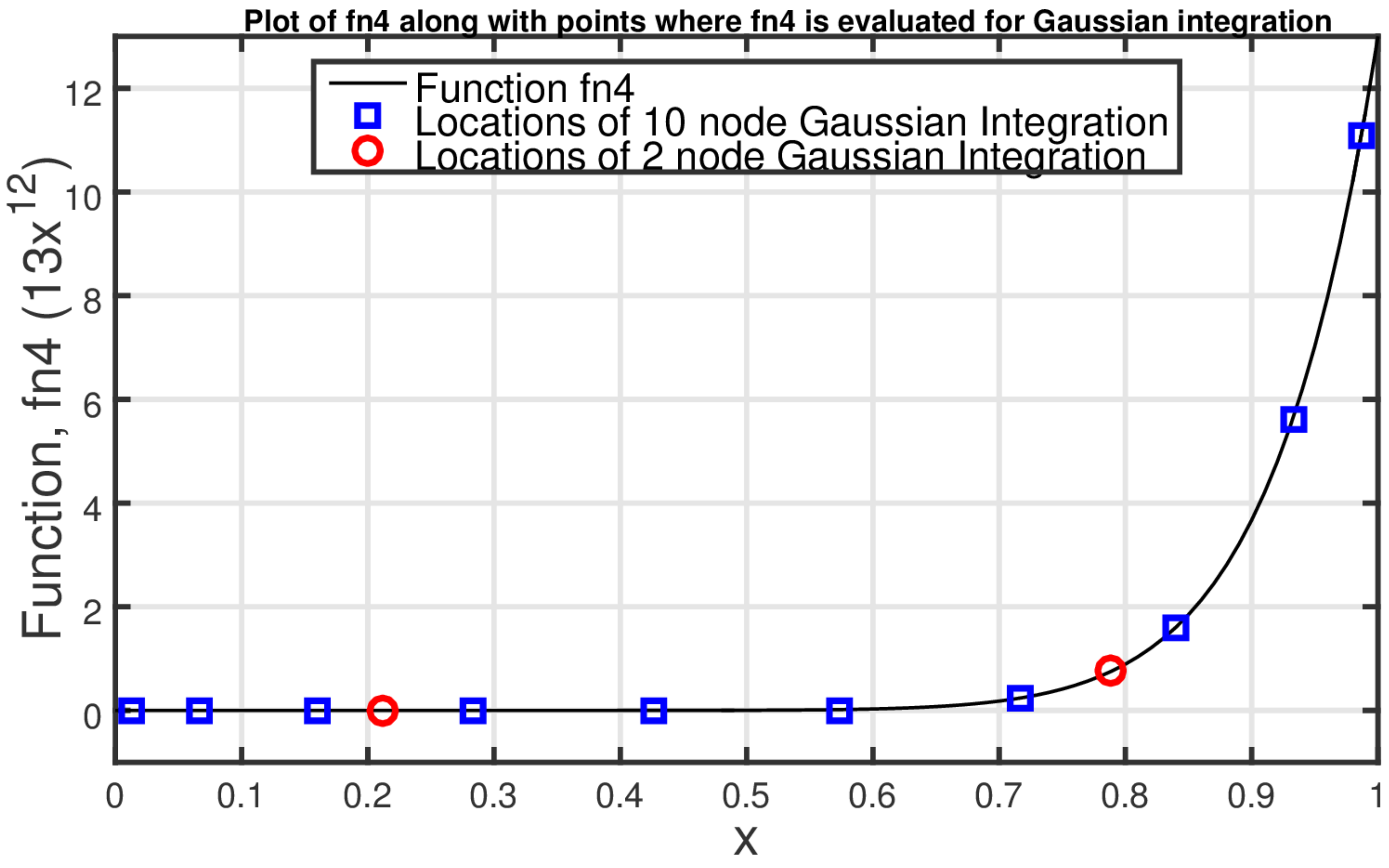
Gaussian Integration

Was looking for an answer with a specific reference to plot, such as “because of the location of the 2 nodes, the rapid rise of the function is not captured”; “the 10 nodes sample the function at enough locations to capture the rapid rise in value, near X = 1”



14 Oct 2016

```
 subroutine fn1(y,c)
       implicit double precision(a-h,o-z)
C
C     Input  : y [place where fn1(x) is to be evaluated]
C     Output : c [value of fn1(x) for input value of y,
C                 using an appropriate variable
C                 transformation (see below) ]
C
C     fn1(x) = 3x^2
C
C     y is the variable used for Gaussian integration;x is the original variable
C     The integral of the function is needed from x = 0. to 1.
C     However, Gaussian integration by definition runs from y = -1. to 1.
C
C     --------------------------------------------
C     1. a relation that describes x as a fn of y
C     --------------------------------------------
C      In Gaussian Quadrature, the x varible in f(x) will be
C      substituted by g(y)=((b-a)/2)*y+(b+a)/2.
C
C     -----------------------------------
C     2. a relation that describes dx/dy
C     -----------------------------------
C     dx/dy denotes the relationship between x and y, where is
C     approximate to ((b-a)/2)*ci. ci is the Gauss coefficient.
C
C     The total equation of Gaussian Quadrature would be:
C
C     ((b-a)/2)*Sigma(ci*f(g(y)))=((b-a)/2)*Sigma(ci*f(((b-a)/2)*y+(b+a)/2))
C
C     In this function fn1, the calculaters:
C       x= g(y) = ((b-a)/2)*y+((b+a)/2)
C       dx_dy= ((b-a)/2)
C       output c=f(g(y))*dx_dy
C
C     Note: because x integrate from 0 to 1, (b-a)/2=(b+a)/2=1/2
C           weight(i)=ci is given in main program.
C ================================================================
       x=0.5*y+0.5
       dx_dy=0.5
       c=3.d0*(x**2)*dx_dy

       return
       end
```

```
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       return
       end
```

Outstanding commenting!

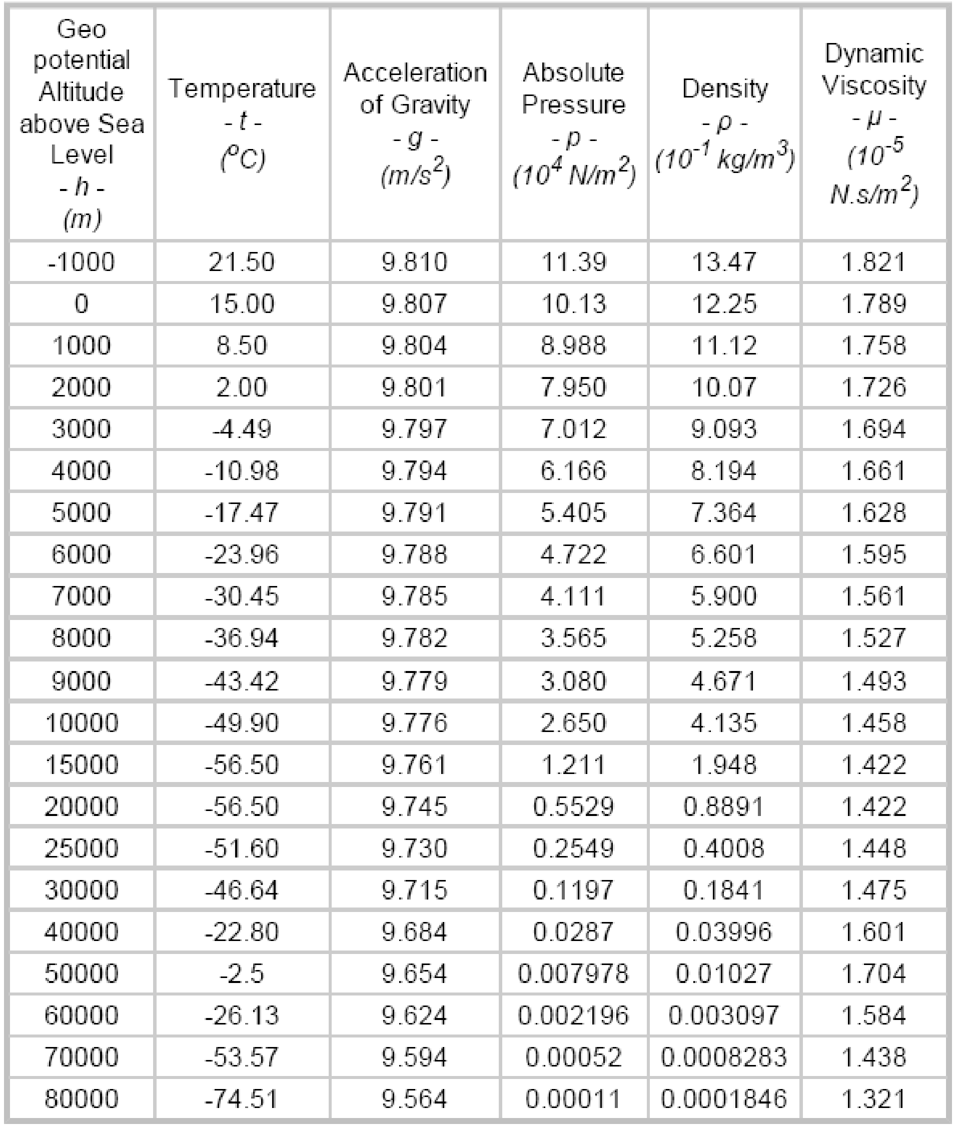
Comments are so very helpful, to you and others, and good commenting of code is vitally important for group research efforts

AOSC 652: HW 06, Part 1 Points Subtracted

No reference to quantitative errors from Simpson’s rule and/or precision of Gaussian integration, in reply for which we would choose	Up to 3
Explanation of why 10-node works better than 2-node, couched in generalities rather than specifics	Up to 5
Attempt to explain Gaussian integration, as if operated like the Trapezoidal Rule	Up to 5
Problem with code used to determine where nodes should be placed	Up to 5
Problem with plot	Up to 10

AOSC 652: Analysis Methods in AOSC

US Standard Atmosphere



Geo potential Altitude above Sea Level - h - (m)	Temperature - t - (^oC)	Acceleration of Gravity - g - (m/s^2)	Absolute Pressure - p - $(10^4\ N/m^2)$	Density - ρ - $(10^{-1}\ kg/m^3)$	Dynamic Viscosity - μ - $(10^{-5}\ N.s/m^2)$
-1000	21.50	9.810	11.39	13.47	1.821
0	15.00	9.807	10.13	12.25	1.789
1000	8.50	9.804	8.988	11.12	1.758
2000	2.00	9.801	7.950	10.07	1.726
3000	-4.49	9.797	7.012	9.093	1.694
4000	-10.98	9.794	6.166	8.194	1.661
5000	-17.47	9.791	5.405	7.364	1.628
6000	-23.96	9.788	4.722	6.601	1.595
7000	-30.45	9.785	4.111	5.900	1.561
8000	-36.94	9.782	3.565	5.258	1.527
9000	-43.42	9.779	3.080	4.671	1.493
10000	-49.90	9.776	2.650	4.135	1.458
15000	-56.50	9.761	1.211	1.948	1.422
20000	-56.50	9.745	0.5529	0.8891	1.422
25000	-51.60	9.730	0.2549	0.4008	1.448
30000	-46.64	9.715	0.1197	0.1841	1.475
40000	-22.80	9.684	0.0287	0.03996	1.601
50000	-2.5	9.654	0.007978	0.01027	1.704
60000	-26.13	9.624	0.002196	0.003097	1.584
70000	-53.57	9.594	0.00052	0.0008283	1.438
80000	-74.51	9.564	0.00011	0.0001846	1.321

Source: http://www.engineeringtoolbox.com/standard-atmosphere-d_604.html#

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US Standard Atmosphere

Geo potential Altitude above Sea Level - *h* - *(m)*	Temperature - *t* - *(°C)*	Acceleration of Gravity - *g* - *(m/s²)*	Absolute Pressure - *p* - *(10⁴ N/m²)*	Density - *ρ* - *(10⁻¹ kg/m³)*	Dynamic Viscosity - *μ* - *(10⁻⁵ N.s/m²)*
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Almost certainly, T was specified as a function of altitude, and then pressure was found by integrating the hydrostatic equation.

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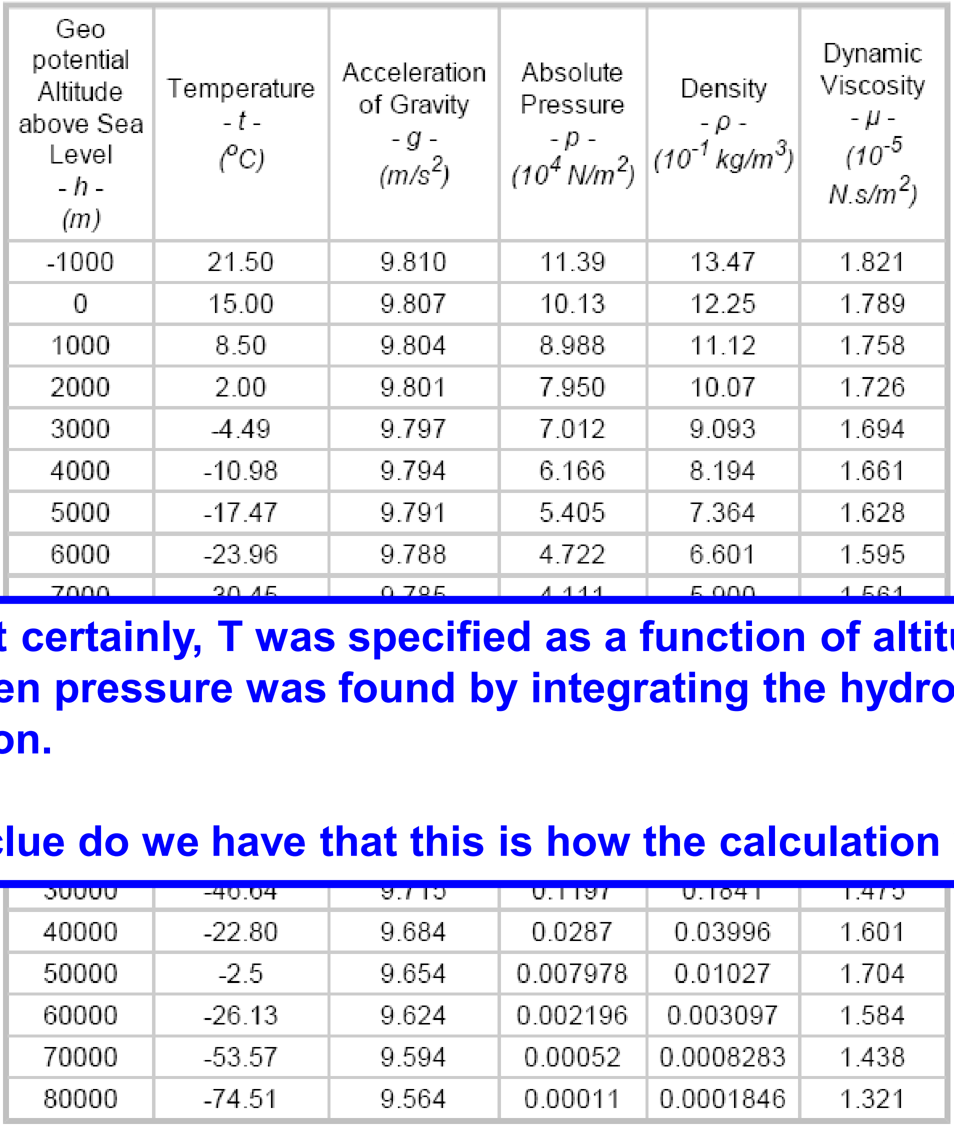
US Standard Atmosphere



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Almost certainly, T was specified as a function of altitude, and then pressure was found by integrating the hydrostatic equation.

What clue do we have that this is how the calculation was done ?

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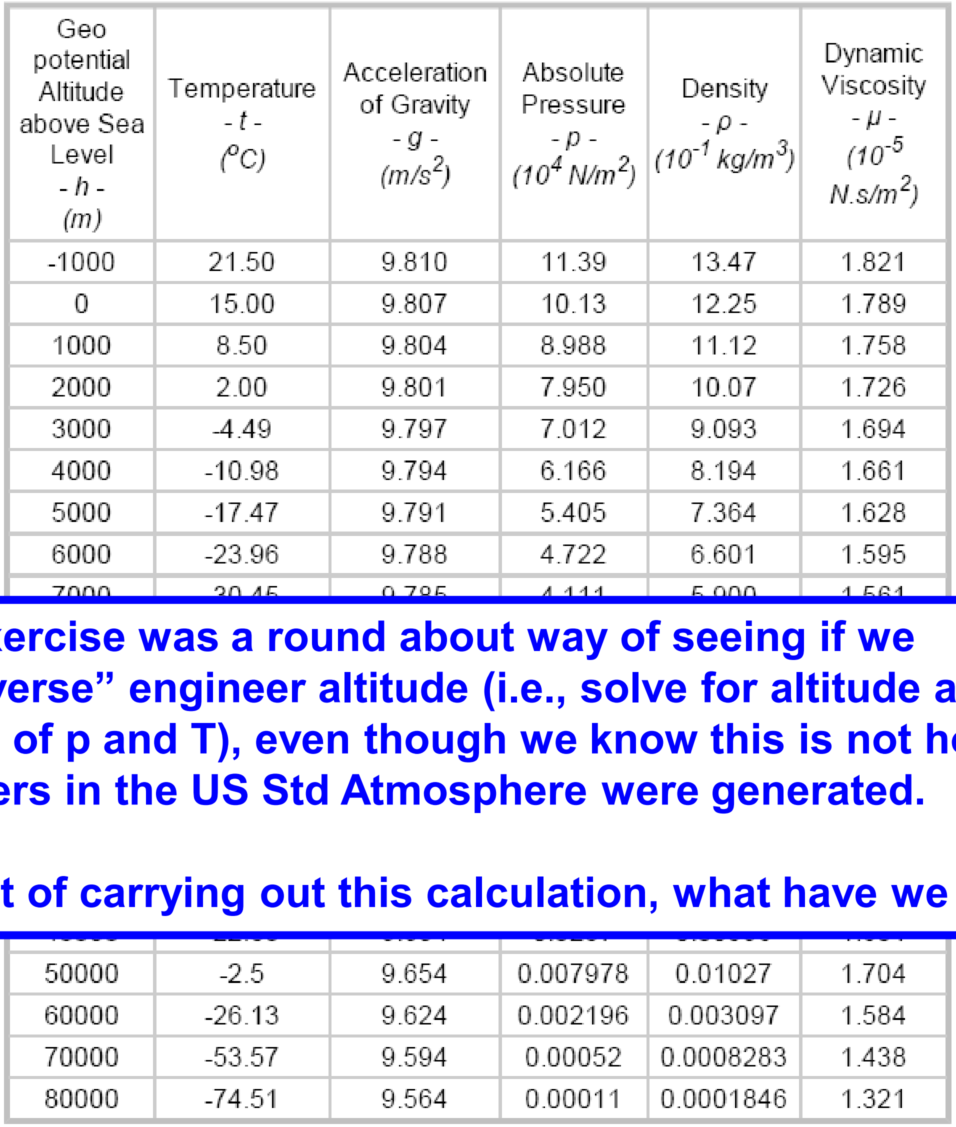
US Standard Atmosphere



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So, our exercise was a round about way of seeing if we could “reverse” engineer altitude (i.e., solve for altitude as a function of p and T), even though we know this is not how the numbers in the US Std Atmosphere were generated.

As a result of carrying out this calculation, what have we learned ?

Source: http://www.engineeringtoolbox.com/standard-atmosphere-d_604.html#

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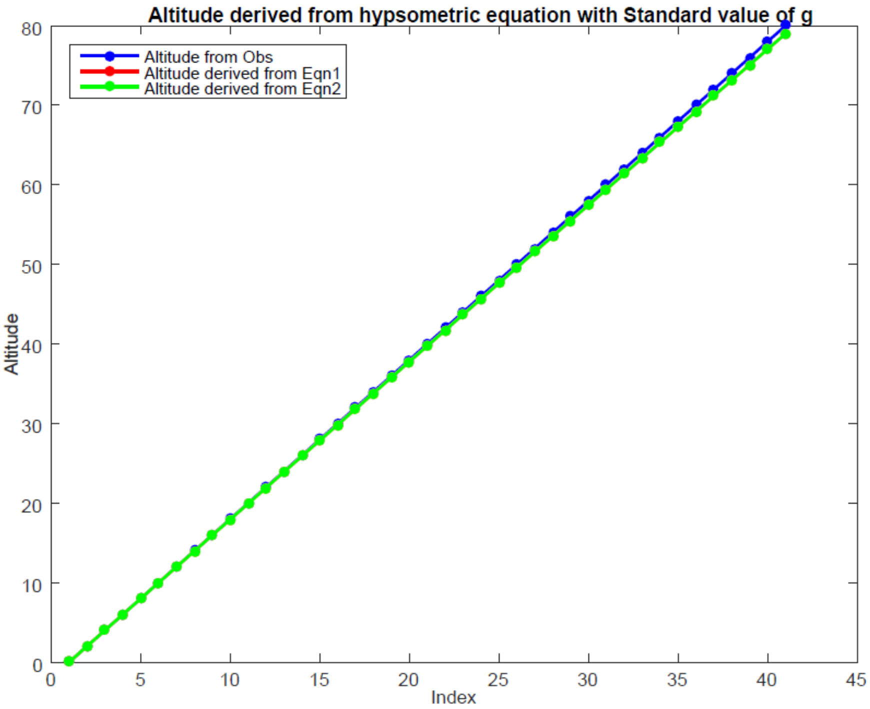
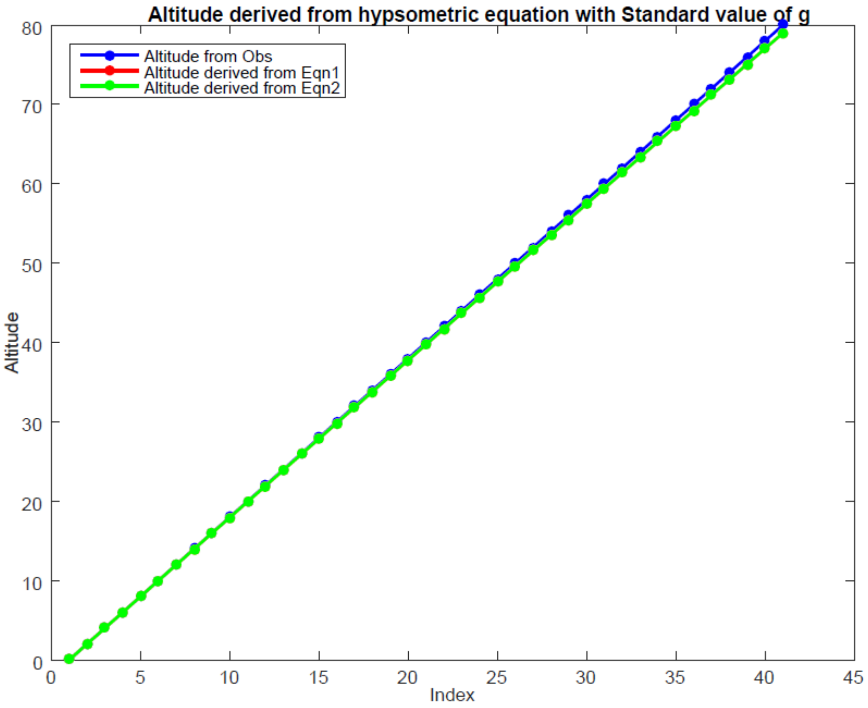

Altitude derived from hypsometric equation with Standard value of g


Altitude derived from hypsometric equation with Standard value of g

The trapezoidal rule, despite “disdain” from numerical methods folks, works rather well (but not perfectly)

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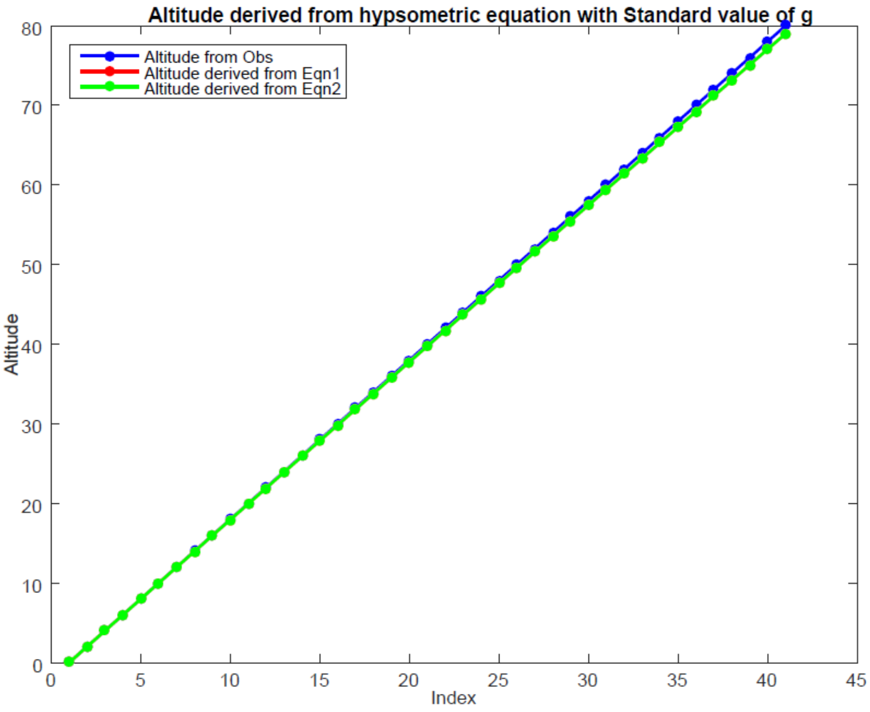

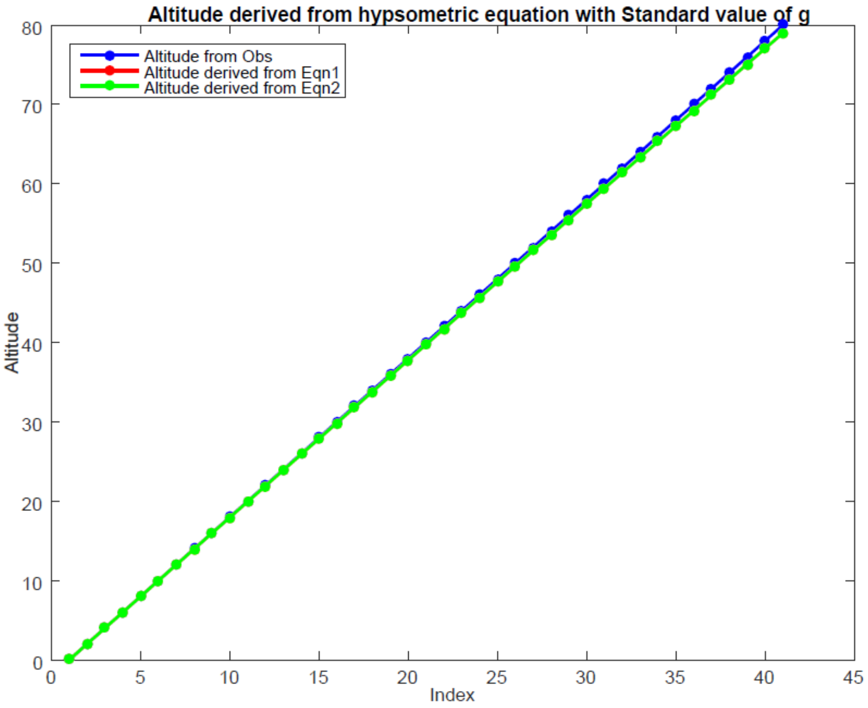

If we want high precision, can not move “g” outside of the integral sign, as is often done in textbooks.

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US Standard Atmosphere

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Why could we not use Simpson's rule ?

Source: http://www.engineeringtoolbox.com/standard-atmosphere-d_604.html#

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Why could we not use Simpson's rule ?

Because points were not evenly spaced with respect to the integrating variable, which in our case is ln(p)

Source: http://www.engineeringtoolbox.com/standard-atmosphere-d_604.html#

Numerical Integration

Simpson's Rule:

$$\int_{x_1}^{x_N} f(x)\,dx \approx \sum_{i=1,\,3,\,5}^{N-2} \frac{x_{i+2}-x_i}{2}\,\frac{f(x_{i+2})+4f(x_{i+1})+f(x_i)}{3}$$

Classic method will work only for *evenly spaced grid* and odd N

Simpson's rule requires there be an odd number of points and that these odd number of points be evenly spaced!

AOSC 652: HW 06, Part 2 Points Subtracted

Not fixing my ***intentional*** mistake, regarding number of header lines	3
Not being specifically quantitative, when comparing results found using our integration to the expected value. Note: “slight difference” just doesn’t cut it in the real world. 0.2 km over-estimate at 80 km much better	Up to 5
Not understanding that Simpson’s Rule requires integrating interval to be evenly spaced	Up to 5
Coding error, calculation of zintegral	Up to 10

Output of original code

```
2,3
Index,Altitude
Estimate of altitude based on the evaluation of the integral of
R/grav Temperature d ln (pressure), for data read from file atm.us45std.dat
 1    0.00
 2    2.00
 3    4.00
 4    5.99
 5    7.99
 6    9.99
 7   11.98
 8   13.97
```

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Correct entry for number of header lines, either fixed by hand (a few students) or in Fortran (a few other students)

```
2,4
Index,Altitude
Estimate of altitude based on the evaluation of the integral of
R/grav Temperature d ln (pressure), for data read from file atm.us45std.dat
 1    0.00
 2    2.00
 3    4.00
 4    5.99
 5    7.99
 6    9.99
 7   11.98
 8   13.97
```

AOSC 652: HW 06, Part 2 Points Subtracted

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Numerical Integration

Simpson’s Rule:

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Classic method will work only for *evenly spaced grid* and odd N

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Numerical Integration

Simpson’s Rule:

$$\int_{x_1}^{x_N} f(x)\,dx \approx \sum_{i=1,\ 3,\ 5}^{N-2} \frac{x_{i+2} - x_i}{2} \frac{f(x_{i+2}) + 4f(x_{i+1}) + f(x_i)}{3}$$

Classic method will work only for *evenly spaced grid* and odd N

Simpson’s rule requires there be an odd number of points and that these odd number of points be evenly spaced!

**How did we implement a version of Simpson’s rule that works for either even or odd number of points ?
(they still must be evenly spaced!)**

Extra credit:

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We formed two branches.
Branch 1: if # of points is odd, use Simpson’s rule
Branch 2: if # of points even, use Simpson’s rule for N-1 of the intervals, and use Trapezoidal rule for remaining interval

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Why do we compare diff1 and diff2?

diff1 and diff2 represent the steepness of the slope at the respective ends of the function.

The trapezoidal rule is evaluated for the end interval with the smallest slope, so that the error in using the Trapezoidal rule will have the least impact.

AOSC 652: Analysis Methods in AOSC

Between now and Monday:

IDL Tract:

Please read chapters 1, 2, and 3 of Bowman

An Introduction to Programming with IDL

Python Tract:

Please read *, *, * of DeCaria,

Python Programming and Visualization for Scientists