Analysis Methods in Atmospheric and Oceanic Science

AOSC 652

Fourier Analysis

Week 9, Day 1

24 Oct 2016

Fourier Series and Spectral Analysis

Any function can be expressed as a sum
of a series of sin and cosine curves:

$$f(t) = \sum_{n=1}^{\infty} a_n \sin(2\pi nt) + b_n \cos(2\pi nt) \qquad (1)$$

f(t) is a "signal" in the time domain,
a_n and b_n are unknown coefficients, and
n has units of 1/time corresponding to the various frequencies of the wave

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n has units of 1/time corresponding to the various frequencies of the wave

Often, you’ll see the expression for the Fourier Transform written quite differently from the above equation

Fourier Series and Spectral Analysis

For example, equation 12.0.1 of Press *et al*. states:

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Or:

$$f(t) = \int_{-\infty}^{\infty} H_{\mathrm{REAL}}(f) \cos(2\pi f t) + i\, H_{\mathrm{IMAGINARY}}(f) \sin(2\pi f t)\, df \qquad (5)$$

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This expression used in Muller and MacDonald reading !

$$f(t) = \int_{-\infty}^{\infty} H_{\mathrm{REAL}}(f) \cos(2\pi f t) + i\, H_{\mathrm{IMAGINARY}}(f) \sin(2\pi f t)\, df \qquad (5)$$

Fourier Series and Spectral Analysis

There are many ways to compute a_n and b_n in equation (1) :

$$f(t) = \sum_{n=1}^{\infty} a_n \sin(2\pi nt) + b_n \cos(2\pi nt) \qquad (1)$$

or H_{REAL} and $H_{IMAGINARY}$ in equation (5)

$$f(t) = \int_{-\infty}^{\infty} H_{\text{REAL}}(f) \cos(2\pi f\, t) + i\, H_{\text{IMAGINARY}}(f) \sin(2\pi f\, t)\, df \qquad (5)$$

Fourier Series and Spectral Analysis

Here, we will focus on a very simple treatment of (5), that given on page 52 of Muller and MacDonald:

$$f(t) = \int_{-\infty}^{\infty} H_{\text{REAL}}(f)\cos(2\pi f\, t) + i\, H_{\text{IMAGINARY}} \sin(2\pi f\, t)\, df \qquad (5)$$

For a *discrete* time series represented by variable $signal(time)$:

$i.e.,$ there exists $signal(1), signal(2), ..., signal(N)$

and corresponding time elements $time(1),\ time(2),\ ...,\ time(N)$

the values of H_{REAL} and $H_{\text{IMAGINARY}}$ at frequency f are given by:

$H_{\text{REAL}} = 0$

$H_{\text{IMAGINARY}} = 0$

do j=1 to N by 1

$H_{\text{REAL}} = H_{\text{REAL}} + signal(\text{j}) * \cos(2\pi f\ \ time(\text{j}))$

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enddo

Fourier Series and Spectral Analysis

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enddo

Simply put:

- if the data oscillate in phase with a sin wave of given frequency *f*, the various values of $H_{IMAGINARY}$ will reinforce and sum together, leading to a large "summed" value of $H_{IMAGINARY}$
- if the data are not in phase with a sin wave of frequency *f*, various values of $H_{IMAGINARY}$ will tend to cancel and the "summed" value of $H_{IMAGINARY}$ will be small.

Fourier Series and Spectral Analysis

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enddo

Assuming time series is equally spaced at Δt and ranges from t=0 to t=t_{MAX}

- $f_{\mathrm{LOW}} = 1 / t_{\mathrm{MAX}}$ ⇦ lowest frequency of interest
- $f_{\mathrm{NYQUIST}} = 1 / 2\,\Delta t$ ⇦ highest frequency of interest

Fourier Series and Spectral Analysis

This is a good time to look at the equation, and get a sense of what the Fourier transform does. It takes your data, multiplies it by a sine wave, and then sums the results. If the data oscillate in phase with the sine wave, so that they are positive together and negative together, then all the terms in the sum are positive and the Fourier amplitude is large. If they drift into phase and then out of phase, then half of the values in the product will be positive and half will be negative, and the sum will be close to zero. You can also see that the sum is not particularly sensitive to sharp changes in the data (e.g. sudden terminations); it is more sensitive to the bulk behaviour of the data, e.g. are most of the data points positive when the sine wave is positive?

Page 52, Muller and MacDonald

Fourier Series and Spectral Analysis

The total power of a signal is the same whether we compute it in the time or frequency domain (Press *et al*., page 492):

$$Total\ Power \equiv \int_{-\infty}^{\infty} \left[signal(time) \right]^2 \, d(time) = \int_{-\infty}^{\infty} \left[H(f) \right]^2 \, df \quad (6)$$

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Often one wants to know "how much power" is contained in the frequency interval between f and $f + df$. In such circumstances, we usually do not distinguish between positive and negative values of f, but rather regard f as varying between 0 (no frequency, or D.C.) and infinity:

$$Power\ (f) \equiv [H(f)]^2 + [H(-f)]^2 \qquad 0 \le f \le \infty \qquad (7)$$

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If *signal(time)* is a real function, the two terms above can be shown to be equal and:

$$Power\ (f) = 2\,[H(f)]^2 \qquad 0 \le f \le \infty \qquad (8)$$

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Press *et al*. (pages 492 & 493) discuss this factor of 2 and how it is sometimes not used. We'll work with normalized power spectrum, such that maximum value is by definition unity, so we will also not consider this factor of 2.

Fourier Series and Spectral Analysis

Returning to our heuristic code:

For a <u>*discrete*</u> time series represented by variable $signal(time)$:

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enddo

$\underline{Power = H_{\text{REAL}}^{\ 2} + H_{\text{IMAGINARY}}^{\ 2}}$ (see page 52, Muller & MacDonald)

Fourier Series and Spectral Analysis

Let's conduct some power spectrum analyses:

Copy files ~rjs/aosc652/week_09/tone*.dat
& ~rjs/aosc652/week_09/fourier_analysis.f

to you work area.

File tone01.dat contains a *signal* vs *time*, for *time* = 0 to 0.25 sec

First, using either hppltd, Python, or IDL produce a plot of signal vs time:

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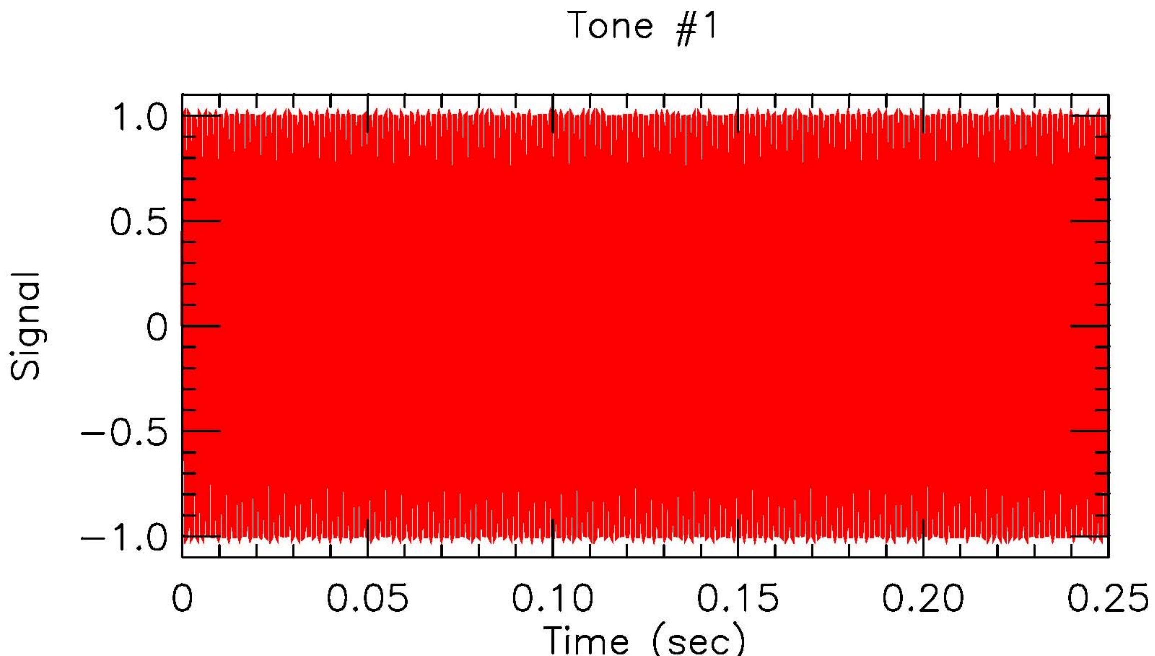

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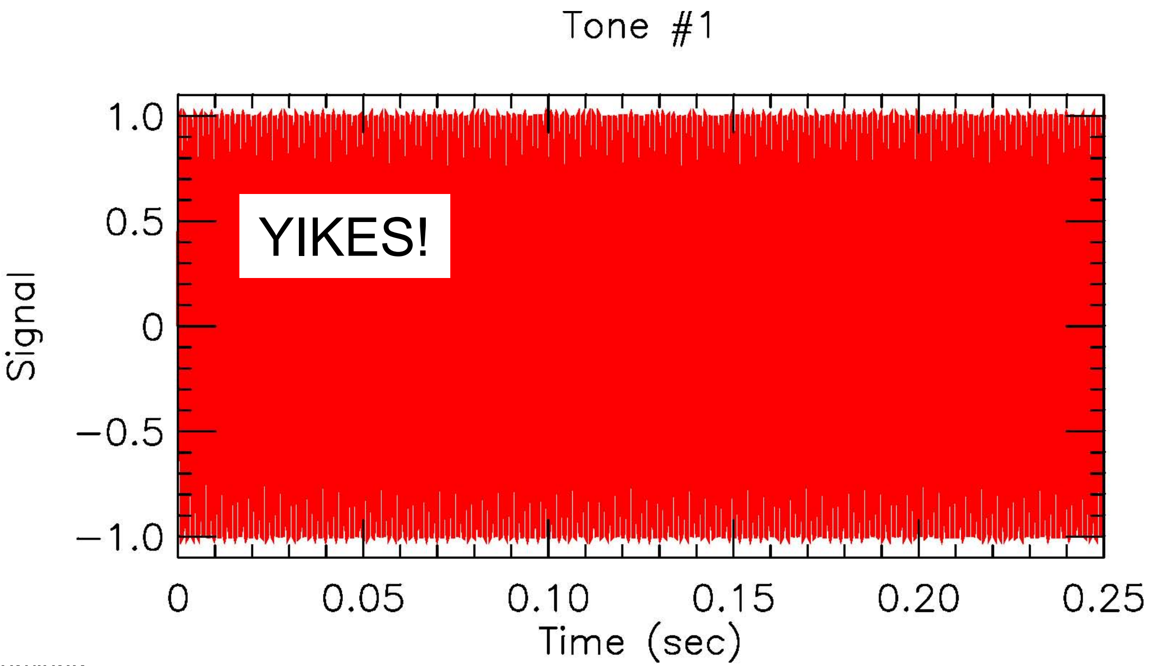

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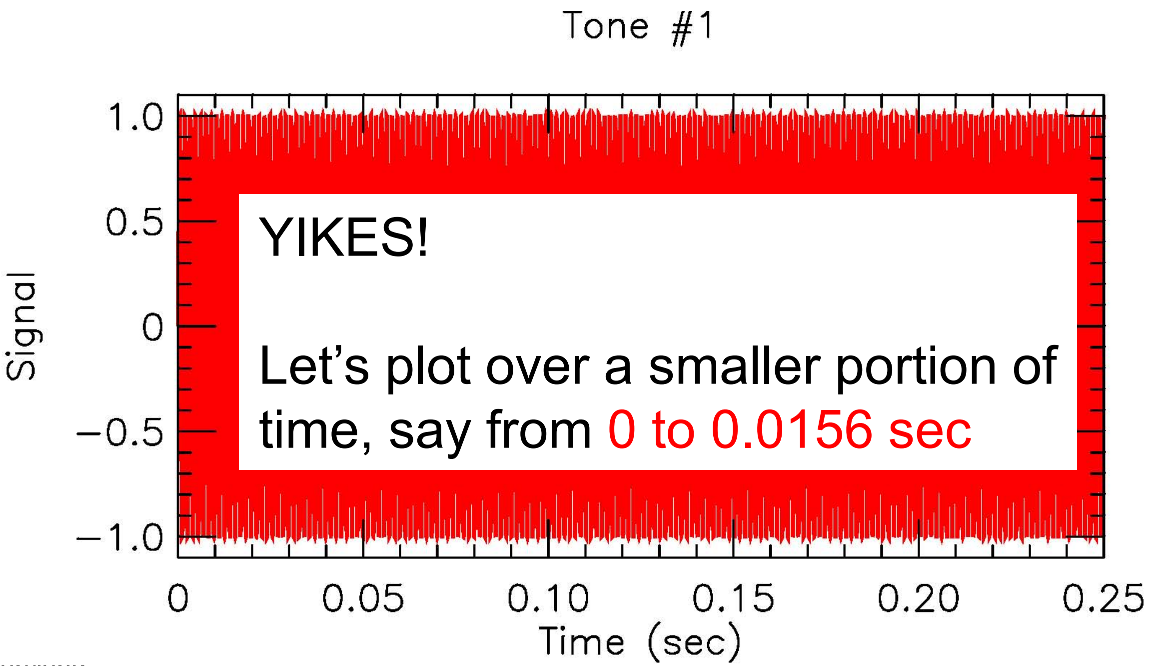

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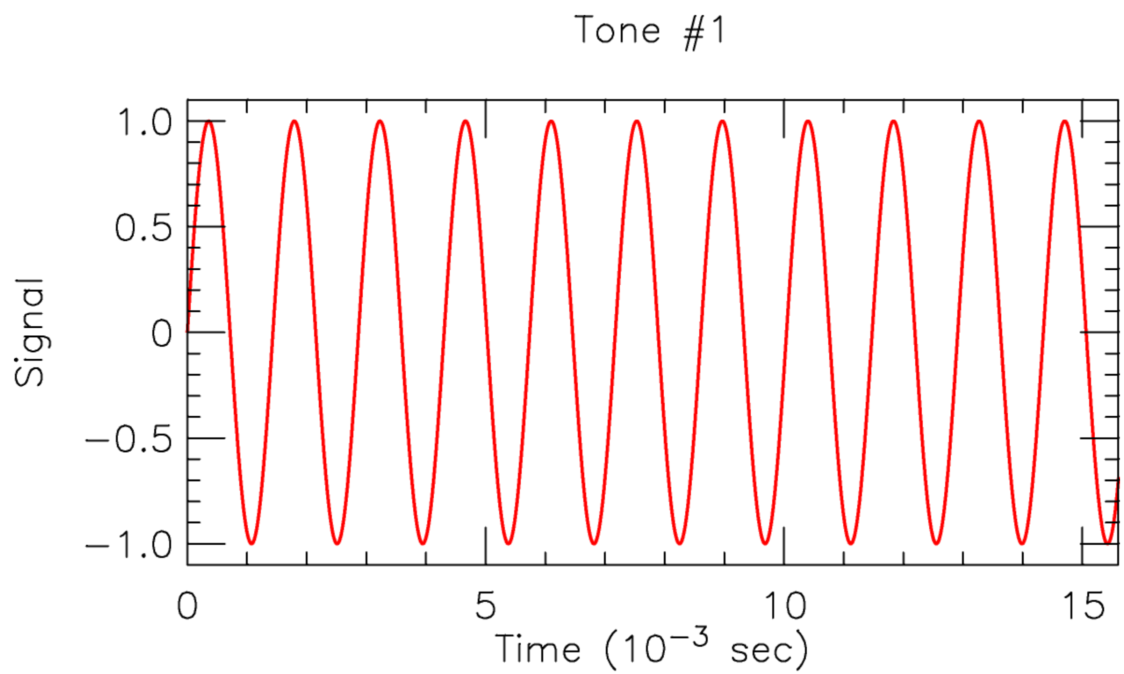

Fourier Series and Spectral Analysis

Compile program fourier_analysis.f and compute power spectrum of tone01.dat, from frequencies of 600 to 1300 Hz, every 1 Hz

Plot the result from 600 to 1300 Hz:

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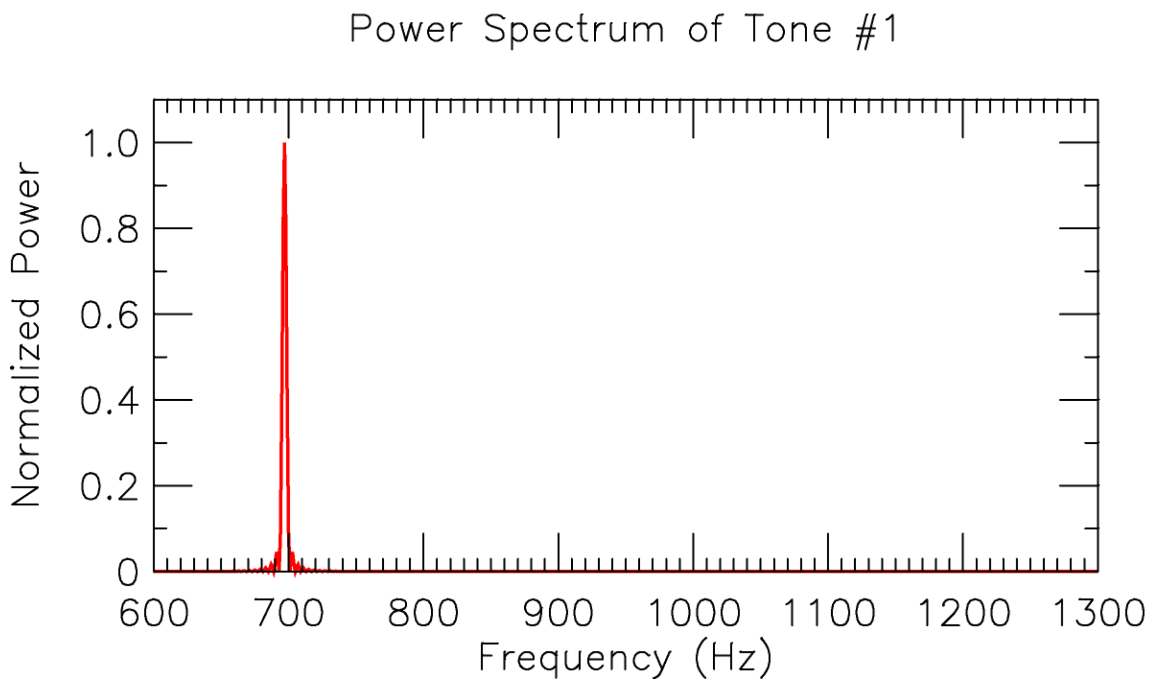

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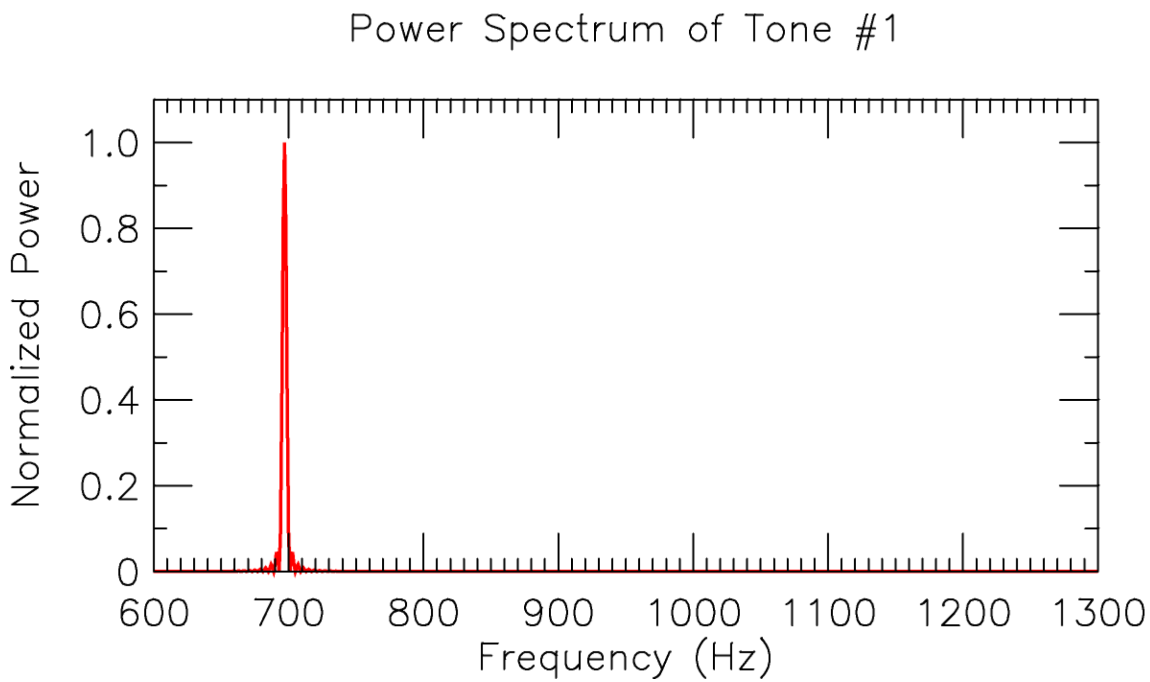
Plot the result from 600 to 1300 Hz:



tone01.dat generated with FORTRAN program ~rjs/aosc652/week_09/tone01_gen.f

Fourier Series and Spectral Analysis

Plot tone02.dat, from 0 to 0.0156 sec :

Fourier Series and Spectral Analysis

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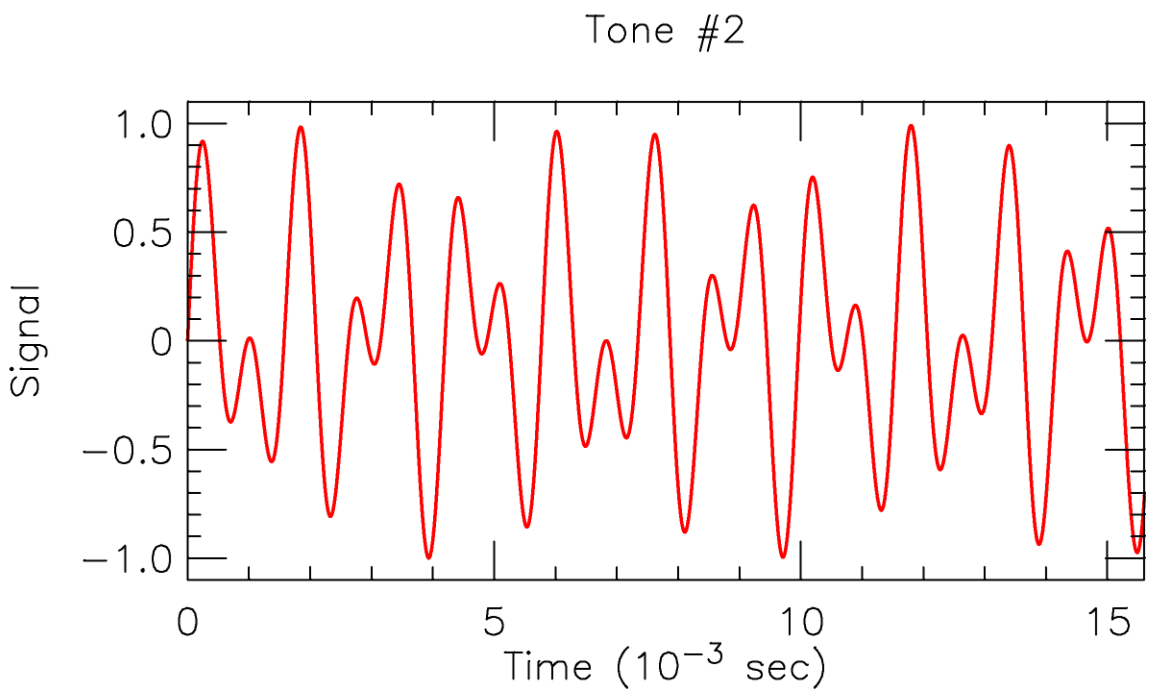
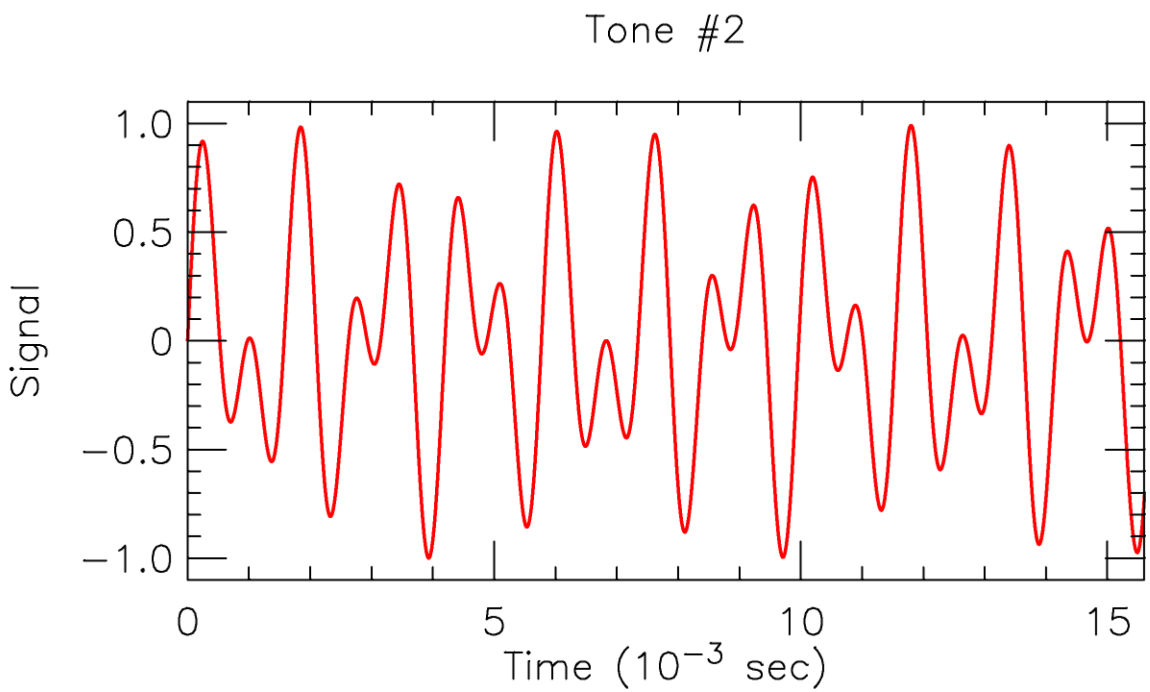


tone02.dat generated with FORTRAN program ~rjs/aosc652/week_09/tone02_gen.f

Fourier Series and Spectral Analysis

Plot tone02.dat, from 0 to 0.0156 sec :



Does this look familiar?

If so, what is it?

Fourier Series and Spectral Analysis

Compute power spectrum of tone02.dat, from frequencies of 600 to 1300 Hz, every 1 Hz

Plot the result from 600 to 1300 Hz:

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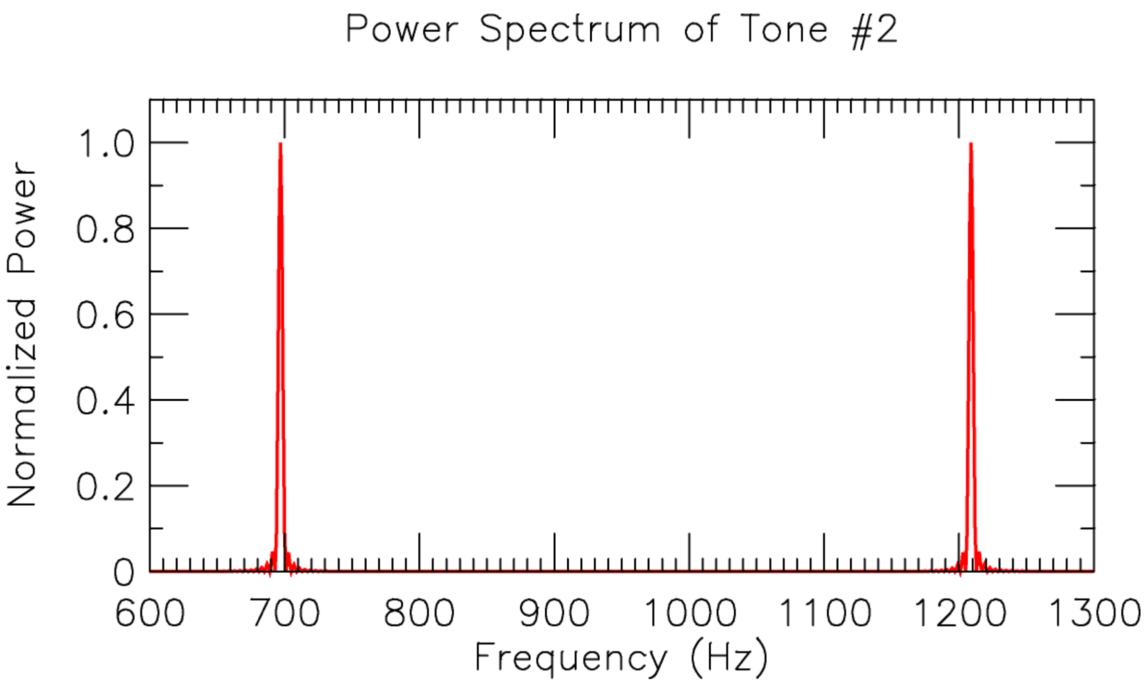

Fourier Series and Spectral Analysis

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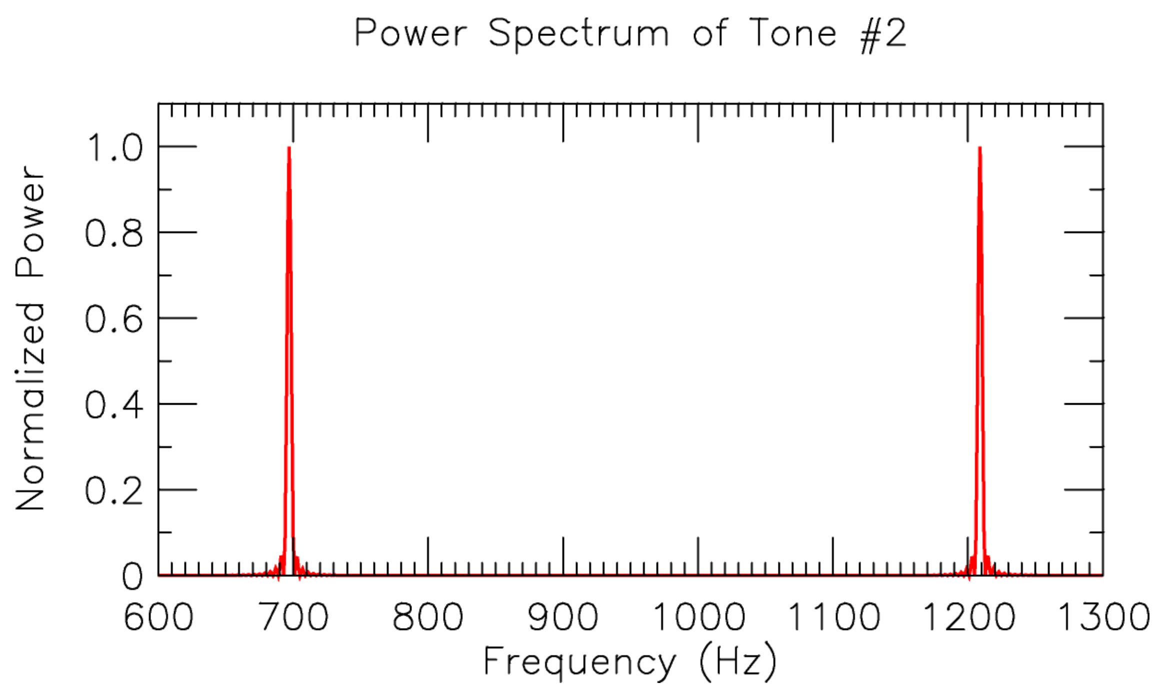
Plot the result from 600 to 1300 Hz:



tone02.dat generated with FORTRAN program ~rjs/aosc652/week_09/tone02_gen.f

Fourier Series and Spectral Analysis

Mini-assignment for Wed:

Solar activity/sunspots (time scale of decades)

Copy file ~rjs/aosc652/week_09/sunspot_number_monthly.dat to your work area.

This file contains *Sunspot Number* vs *Time*, from 1749 to present

Using program fourier_analysis.f, find the Power Spectrum of this time series, for time periods from **7 to 15 years** every **0.1 years** and:

a) display the results
b) answer a few questions about the results

Note: Power Spectrum must be found using the instructor provided FORTRAN program. Plots can be produced using either hpptld, MATLAB, Python, or IDL

Mini-assignment due **start of class on Wed**: no exceptions

Fourier Series and Spectral Analysis

Readings for Wed:

Everyone: pages 63 to 73 of Muller and MacDonald

Team Python: Chapter 18 of DeCaria

Team IDL: Chapter 25 of IDL

We'll all meet here at the start of class on Wed;
we'll split about 20 to 30 mins into class