# Analysis Methods in Atmospheric and Oceanic Science

## AOSC 652

## Ordinary Differential Equations
## Week 12, Day 1

14 Nov 2016

# AOSC 652: Analysis Methods in AOSC

## Student projects:

- 20% of the final grade: you will receive a numerical score for the project and final grade will be found via:

  **Final Grade = (0.1)×(Attendance + Participation) + (0.7)×(Homework) + (0.2)×(Final Project)**

- 28, 30 Nov , 2, 5, and 7 Dec set aside for “in class” work on your project
- Thurs 8 & Fri 9 Dec: *students present their project (10 minute talks)*, prepared using either Powerpoint, Open Office, etc and converted to PDF prior to the start of class
- Each student must turn in a ***brief*** written description of the project as well as all *code* used to complete the project
- Good to begin thinking about your project: application of techniques learned in class to a ***scientific problem of your interest***
- I am available to discuss potential projects at any time

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**⇒ Project Description due this Friday, at start of class**

## Introduction to Differential Equations

- Differential Equations are central to Atmospheric and Ocean Sciences

- They provide *quantitative* descriptions of:
  - motions of air and water ⇨ circulation
  - interaction of light and molecules ⇨ photolysis frequencies and heating rates
  - abundance of gases and particles ⇨ biogeochemical cycling of carbon, nitrogen, sulfur, etc
    ⇨ air quality
    ⇨ ozone depletion and recovery
    ⇨ clouds, aerosols, and precipitation
  - interaction of all of above: ⇨

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    ⇨ air quality
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    ⇨ clouds, aerosols, and precipitation
  - interaction of all of above: ⇨ climate change (impact of human activity on the physical, chemical, and thermodynamic state of the atmosphere and oceans)

## Introduction to Differential Equations

Differential Equation:

An equation that defines the relationship between an unknown quantity and one or more of its derivatives

What is an “ordinary” differential equation?

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What is an "ordinary" differential equation?

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We'll now review examples of "ordinary" differential equations from classical physics and chemistry

## Introduction to Differential Equations

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What is $\tau$ ?

What must be the units of $\tau$ ?

## Introduction to Differential Equations

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Other ways of writing same equation:

$$ {}^{14}\dot{\mathrm{C}} = -\frac{1}{\tau}\ {}^{14}\mathrm{C} $$

$$ {}^{14}\mathrm{C}' = -\frac{1}{\tau}\ {}^{14}\mathrm{C} $$

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$${}^{14}\mathrm{C}' = -\frac{1}{\tau}\ {}^{14}\mathrm{C}$$

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Newton's Second Law of Motion:

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Newton's Second Law of Motion:

*F* = *m a* : the net force on an object is equal to the mass of the object multiplied by its acceleration

How do we express this mathematically as a differential eqn?

$$F\ (t) = m\ \frac{\mathrm{d}^2 x}{dt^2}$$

## Introduction to Differential Equations

Carbon-14 decay: $$\frac{d\ ^{14}\mathrm{C}}{dt} = -\frac{1}{\tau}\ ^{14}\mathrm{C}$$ ← **First Order**

Newton's Second Law of Motion:

$$F(t) = m\frac{\mathrm{d}^2 x}{dt^2}$$ ← **???**

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## Introduction to Differential Equations

Heat Diffusion Equation:

$$\frac{\partial\, T}{\partial\, \mathrm{t}} = \frac{1}{\rho\ c}\ \frac{\partial}{\partial z}\left(\kappa \frac{\partial\, T}{\partial z}\right)$$

What is $T$ ?
What is $t$ ?
What is $\rho$ ?
What is $c$ ?
What is $\kappa$ ?
What is $z$ ?
What is $\kappa\ \delta T/\ \delta z$ ?

## Introduction to Differential Equations

Heat Diffusion Equation:

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c} \frac{\partial}{\partial z} \left( \kappa \frac{\partial T}{\partial z} \right)$$

What is $T$ ? Temperature

What is $t$ ? time

What is $\rho$ ? density

What is $c$ ? specific heat

What is $\kappa$ ? thermal conductivity

What is $z$ ? distance

What is $\kappa\ \delta T / \delta z$ ? conductive heat flux

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| | | **Units** |
|---|---|---|
| What is $T$ ? | Temperature | **(K)** |
| What is $t$ ? | time | **(sec)** |
| What is $\rho$ ? | density | **(kg/m$^3$)** |
| What is $c$ ? | specific heat | **(J kg$^{-1}$ K$^{-1}$)** |
| What is $\kappa$ ? | thermal conductivity | **(J m$^{-1}$ s$^{-1}$ K$^{-1}$)** |
| What is $z$ ? | distance | **(m)** |
| What is $\kappa\, \delta T / \delta z$ ? | conductive heat flux | **(J m$^{-2}$ s$^{-1}$ )** |

## Introduction to Differential Equations

Heat Diffusion Equation:

$$\frac{\partial T}{\partial t} = \frac{1}{\rho c} \frac{\partial}{\partial z} \left( \kappa \frac{\partial T}{\partial z} \right)$$

Sometimes written as:

$$\frac{\partial T}{\partial t} = \alpha \left( \frac{\partial^2 T}{\partial z^2} \right)$$

where $\alpha$ is a constant that describes the rate of heat diffusion

## Solutions of Ordinary Differential Equations

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What is this called?

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Initial Condition

If the independent variable were spatial rather than temporal, what would this term be called?

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Half-life

How much has the amount of $^{14}$C decayed, relative to the initial condition, when $t = \tau$ ?

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Times of $t = \tau$, $t = 2\,\tau$, etc are often called one *e-folding time*, two *e-folding time*, etc.

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What is the half life of $^{14}C$ ?

Given this half life and the fact that all $^{14}C$ measurable in a paleo-climatic biological sample reflects the amount of $^{14}C$ initially present in the sample at the time it was last “living”, minus isotopic decay, over what time horizon range can carbon-14 dating be applied?

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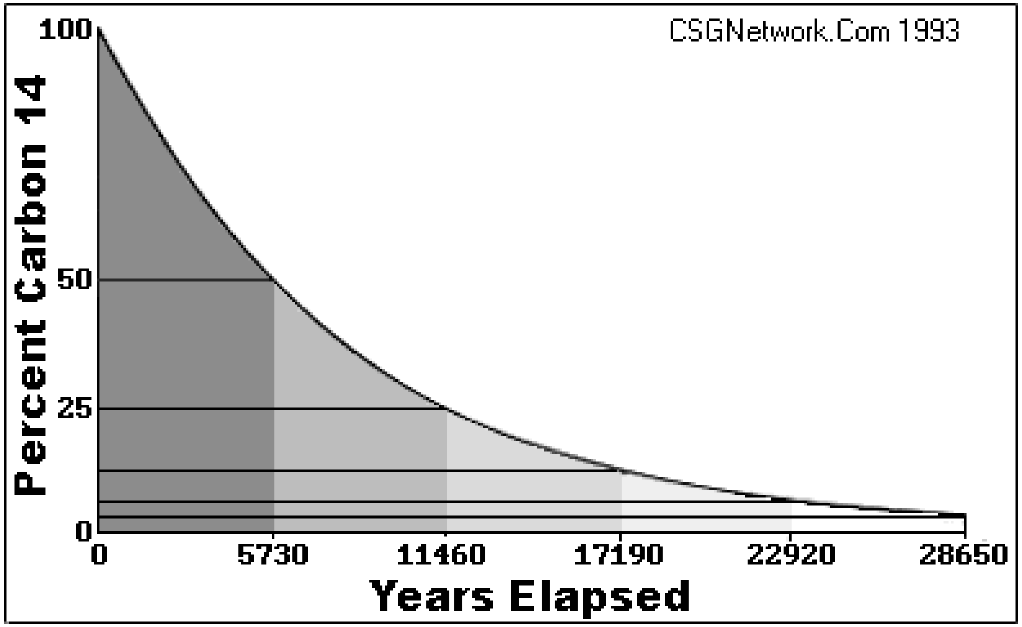


http://www.csgnetwork.com/carbon14datecalc.html

## Introduction to Differential Equations

Beer-Lambert Law:

$$F(z,\lambda) = F_{TOA}(\lambda)\, e^{-\tau(z,\, \lambda)}$$ (TOA : Top of Atmosphere)

where:

$$\tau(z, \lambda) = m \int_{z}^{\infty} \sigma_{\lambda}\, [C]\, dz'$$

$F$ : solar irradiance (photons/cm$^2$/sec)
$\sigma_\lambda$ : absorption cross section (cm$^2$)
$C$ : concentration of absorbing gas (molecules/cm$^3$)
$m$ : airmass: ratio of slant path to vertical path, equal to $1/\cos(\theta)$ for $\theta < \sim 75°$
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**What is this $\tau$ called?**

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What approximation can we make that will allow this equation to be solved analytically?

# AOSC 652: Analysis Methods in AOSC

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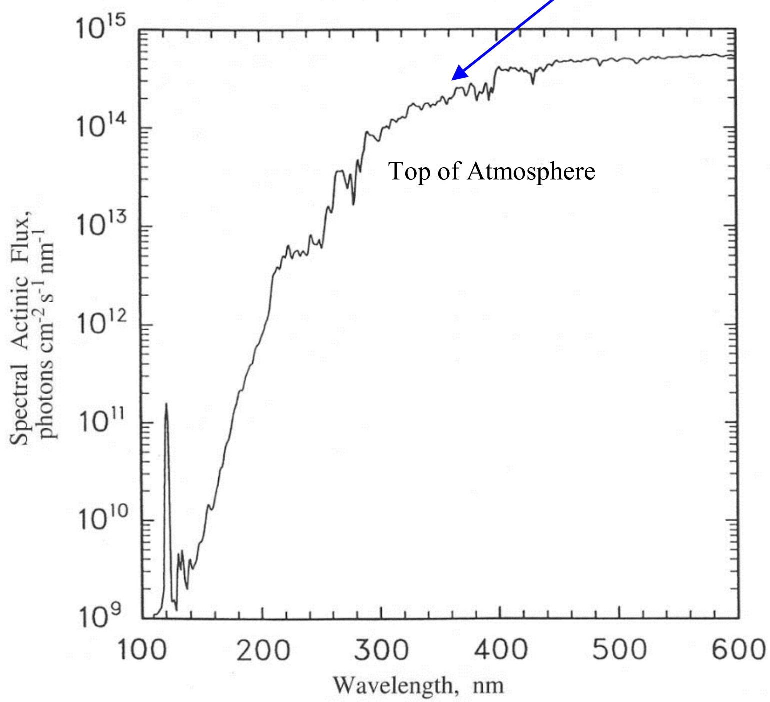
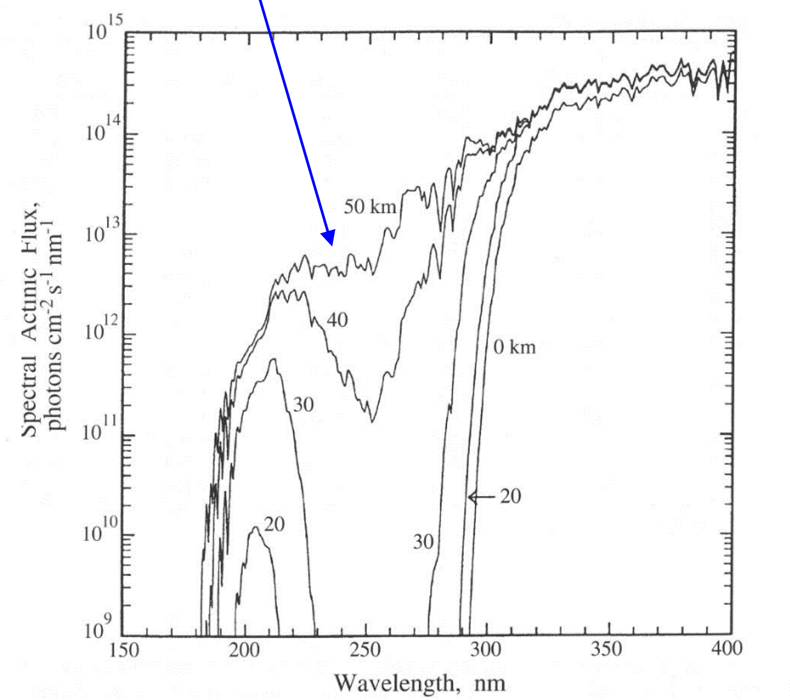
**Controls how atmosphere goes from this …** **to this !**




GURE 3.3 Solar spectral actinic flux (photons $cm^{-2}$ $s^{-1}$ $nm^{-1}$) at various altitudes and at the Earth's surface (DeMore et al., 1994).

From DeMore et al., *Chemical Kinetics and Photochemical Data for Use in Stratospheric Modeling*, Evaluation No. 11, 1994.

From Seinfeld and Pandis, *Atmospheric Chemistry and Physics*, 1998.

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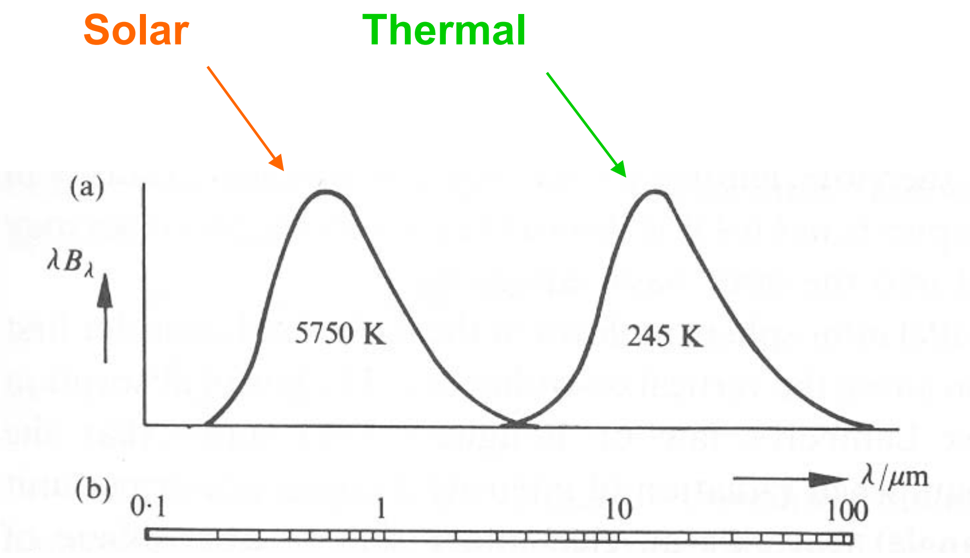
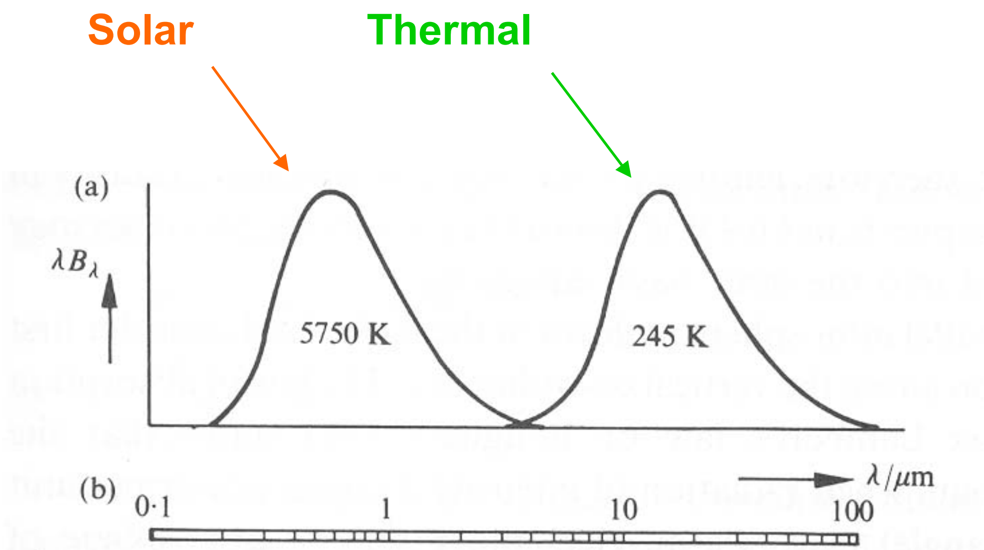



Curves of black-body energy versus wavelength for 5750 K (Sun's approximate temperature) and for 245 K (Earth's mean temperature).
The curves are drawn with equal area since, integrated over the entire Earth at the top of the atmosphere, the solar (downwelling) and terrestrial (upwelling) fluxes must be must be equal.

From Houghton, *Physics of Atmospheres*, 1991

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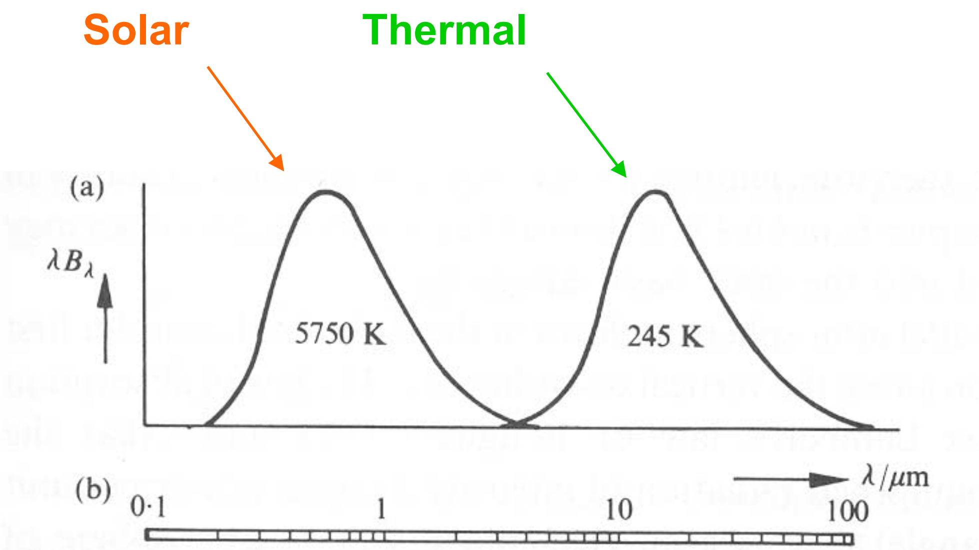


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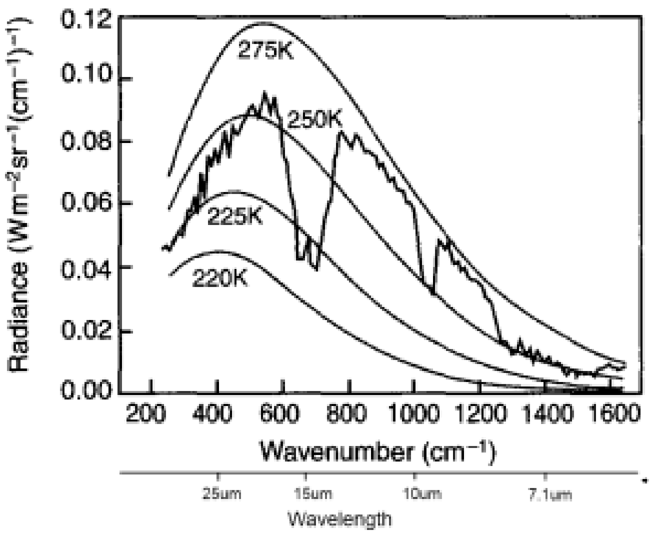


## Solutions of Ordinary Differential Equations

Many simple equations simply can not be integrated to yield an analytic solution. For instance, the simple equation:

$$\frac{dy}{dt} = \frac{y}{1+y^2}$$

can be rearranged and integrated to yield:

$$\ln y + \frac{y^2}{2} = t + C$$

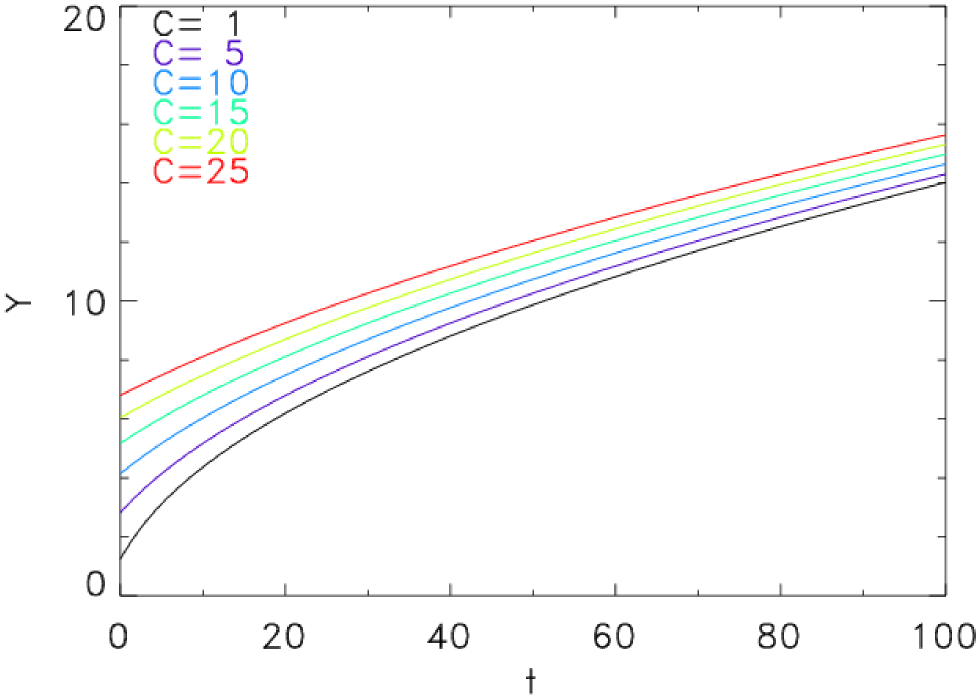
which can be used to describe the evolution of *y(t)* as a function of *t* and *C* : i.e., can generate a family of plots of *y* versus *t*, for various values of *C*.

## Solutions of Ordinary Differential Equations

Solution:

$$\ln y + \frac{y^2}{2} = t + C$$

represented by a family of curves:



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**These plots, together with a measurement of *y* at particular time *t* , provide the "solution".**

**This type of solution is called an *implicit solution***

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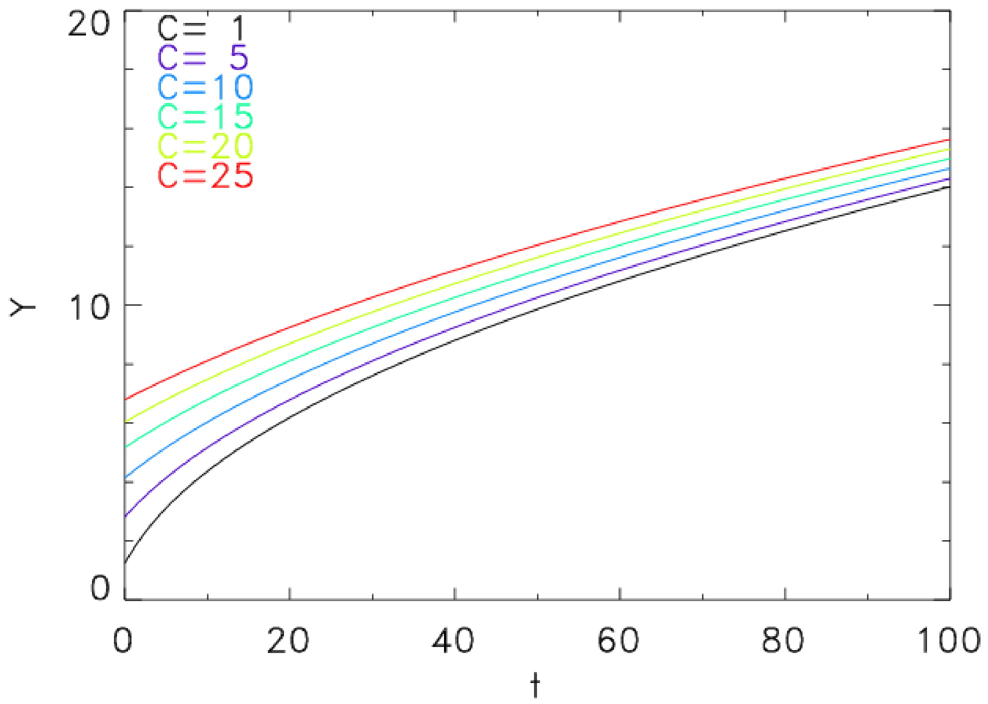


One rather important solution is not yet represented

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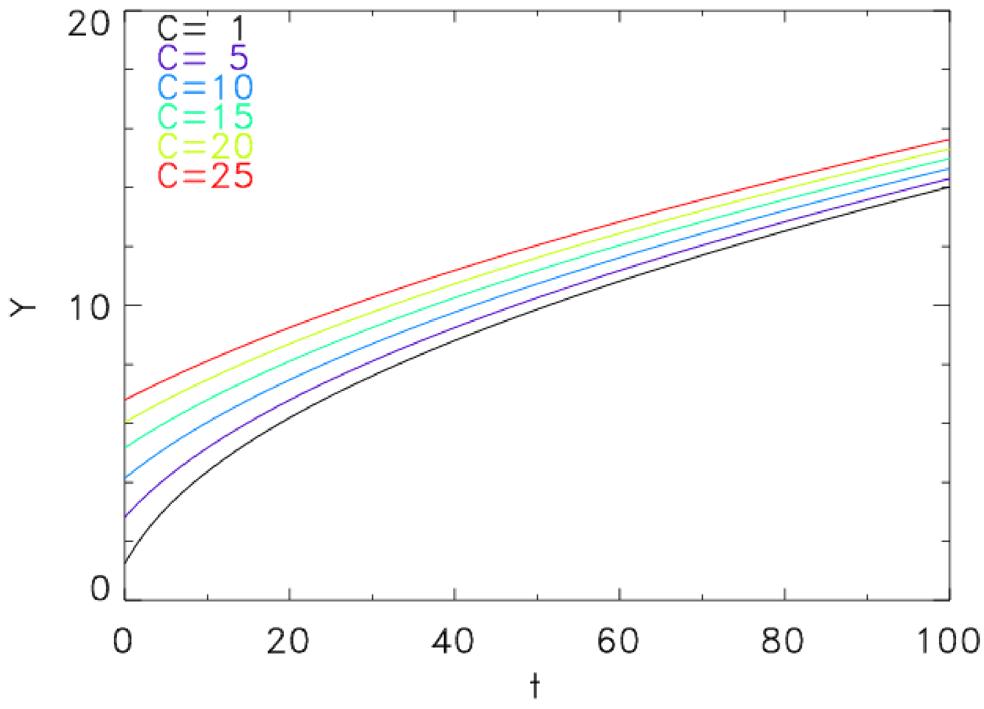


The equilibrium solution, *y* = 0, is not represented by this figure

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Readings for Wednesday:

- 25 pages from Storey
- 7 pages from Press